Submitted to the Brazilian Journal of Probability and Statistics

Optimal variable acceptance sampling based on decision tree method: A Bayesian approach under Type-II censoring

A. M. Mathai[1,a], M. Kumar[*1,b] and J. T. Thomas[2,c]

[1]*Department of Mathematics, National Institute of Technology, Calicut, Pin Code - 673601, India,*
[a]*ashlyn_p190073ma@nitc.ac.in;* [b]*mahesh@nitc.ac.in*

[2]*School of Engineering, University of Wollongong in Dubai, Dubai Knowledge Park, PO Box 20183, UAE,*
[c]*juliat@uow.edu.au*

Abstract. Acceptance sampling plans are essential for quality assurance in reliability testing. This study introduces a novel approach to variable acceptance sampling plans under Type-II censoring, assuming an exponential distribution for failure times with an inverse gamma prior. The proposed model applies a decision tree algorithm, simplifying the process compared to traditional methods that rely on complex nonlinear or stochastic optimization. By using Bayesian inference, the model incorporates prior knowledge and updates decisions with new data, minimizing the number of failures required to terminate the test, while ensuring product reliability and reducing testing costs. The decision tree method with backward induction is used to evaluate costs from the leaf nodes to the root, ensuring optimal decision-making with minimum cost. A real-life case study illustrates the practical applications of this method, and a comparative analysis with an existing model demonstrates that it is not only simpler but also has a significantly higher power of test, making it suitable for real-world applications. A sensitivity analysis study is also conducted using simulated data to assess how changes in cost parameters and prior parameters influence the optimal decision cost.

1 Introduction

Acceptance sampling plans (ASPs) are essential in quality control, determining whether a lot of products meet standards without testing each item. A comprehensive inspection is often impractical due to cost and time, especially, when tests are destructive. Censoring schemes, such as Type-I (fixed time T) (Tsai et al. (2012)) and Type-II (fixed number of failures r) (Chen, Yang and Liang (2016)), address these challenges by reducing the time and expense of testing while maintaining product quality. The primary objective is to find the optimal values of T and r to balance the costs associated with accepting, rejecting, or conducting a comprehensive inspection of a lot. Kumar and Ramyamol (2016) developed an optimal variable ASP using the maximum likelihood estimator for the exponential distribution, incorporating both Type-I and Type-II censoring methodologies. However, advanced censoring schemes, such as hybrid censoring (Balakrishnan and Kundu (2013)), progressive censoring (Balakrishnan and Cramer (2014)), and accelerated life testing, have become increasingly important in addressing more complex and time-sensitive testing environments. Chakrabarty, Roy and Chowdhury (2023) constructed a decision model to determine the optimal sampling plan for the Weibull model under an accelerated life test setting, utilizing a Type-I hybrid censoring scheme for products covered under warranty. Lee et al. (2024) introduced ASPs using progressively Type-II right-censored samples for lifetime testing of exponential lifetime products and evaluated product quality through the process capability index under this censoring scheme. Some examples of recent works include Kumar, Bajeel and Ramyamol

[*]Corresponding Author (email: mahesh@nitc.ac.in)

Keywords and phrases. Acceptance sampling plan, Bayseian estimation, Type-II censoring, exponential distribution, decision tree method.

(2020), Ramyamol and Kumar (2020), Lou, Zheng and Yang (2023), AlSultan and Al-Omari (2023), which explore ASPs using various censoring schemes and accelerated life-testing methods for different distributions.

Bayesian estimation provides a more efficient approach by incorporating prior knowledge and updating it with new data, enabling better decision-making than the conventional methods. Integrating Bayesian methods into ASPs minimizes testing efforts, reduces costs, and protects both producers and customers from unnecessary risk. Earlier studies by Yeh (1994, 1990), and Yeh and Choy (1995) adopted a Bayesian approach to investigate single-variable sampling plans for the exponential distribution, focusing on Type-I, Type-II, and random censoring schemes, respectively, laying the groundwork for the integration of Bayesian methods in ASP design across various censoring techniques. Aminzadeh (2003) developed variable ASPs using an inverse Gaussian model with costs of acceptance, rejection, and inspection. Chen, Liang and Yang (2021) studied a Bayesian sampling plan for two exponential distributions linked by the cumulative exposure model through a simple step-stress accelerated life test under the Type-II censoring scheme. Mathai and Kumar (2024) applied Bayesian estimation with an inverse gamma prior, to design ASPs under Type-I hybrid censoring schemes for the exponential distribution. In recent years, the Bayesian model has been used extensively in designing ASPs by many authors; for instance, Liang and Yang (2011), Prajapati and Kundu (2023), Prajapat et al. (2023) and so on. Apart from variable ASPs, Bayesian estimation methods have been applied in areas like integer-valued time series, as discussed by Peng, Yang and Dong (2024) and Yang et al. (2022).

In all these ASPs, conventional methods of complex nonlinear or stochastic optimization are used, which can be time-consuming and computationally intensive. While the works discussed so far focus on variable ASPs with complex optimization methods, there has been notable research in attribute sampling plans that employ decision tree approaches. With the advent of artificial intelligence in quality control of manufacturing industries, machine learning algorithms are extensively used for decision-making. The decision tree model has been a top-notch decision-making algorithm in computer science and engineering, and this algorithm can also be applied to statistical models. This algorithm offers a simpler and more efficient alternative for determining optimal parameters, bypassing the need for intricate optimization techniques. For instance, Fallahnezhad, Niaki and Vahdat Zad (2012) introduced a revised Bayesian model using a decision tree approach with the backward induction method for decision-making, while Fallahnezhad and Babadi (2015) extended this work by accounting for inspection errors and assuming a beta distribution for the number of defectives. The economic aspects of Bayesian acceptance sampling problems were explored by Fallahnezhad and Aslam (2013), where decision-making was guided by minimizing quality inspection costs along with backward induction. Recently, Thomas and Kumar (2023) developed a model using Bayesian inference for the binomial-Poisson model, paired with a decision tree method and backward induction, to determine the additional reviews needed to find unidentified bugs in software and the expected costs of various outcomes. This provides an insight into practical applications of ASPs using decision trees.

The proposed work introduces a novel approach to variable ASPs by incorporating a decision tree algorithm, setting it apart from existing studies in the literature. While Azam, Aslam and Niaki (2020) have previously developed the prediction decision trees under the repetitive group sampling plan based on process capability index, our focus is on applying the decision tree approach in an optimal Bayesian ASP under a Type-II censoring scheme for exponentially distributed lifetimes with an inverse gamma prior. In our model, Bayesian inference is employed as the decision criterion for accepting or rejecting the lot. The proposed decision tree method operates through an algorithm of a sequence of carefully executed queries or tests, where each outcome influences subsequent tests. The backward induction is applied here to evaluate costs from the leaf nodes of the decision tree and works backward to the root node to identify the optimal set of decisions. Cost functions for corresponding decisions are derived and the optimal number of observed failures for test termination is determined, ensuring a decision on the lot at minimal cost. The structure of the paper is outlined as follows: Section 2 presents a detailed overview of the proposed model, including cost derivation and the decision tree algorithm. In Section 3, the model is demonstrated with a real-life example and simulated data. A sensitivity analysis study is also conducted. Finally, concluding remarks are provided in Section 4.

2 Bayesian modeling in decision tree approach for variable sampling plan

In manufacturing, ensuring the quality of a lot while minimizing costs is a critical challenge, especially when testing is destructive or time-consuming. Bayesian inference offers an effective approach to decision-making under uncertainty by incorporating prior knowledge and updating beliefs as new data becomes available. This approach enables more informed decisions, reducing the need for excessive testing while maintaining product reliability. This study aims to develop an acceptance sampling plan with a decision tree method based on Bayesian inference that determines whether to accept the lot, reject it, or subject it to 100% testing. By using Bayesian methods, the proposed model aims to minimize testing costs and inspection efforts while balancing risks for both producers and customers, optimizing the quality assurance process.

Consider a sample of size n drawn from a lot containing N units, where the failure times of all units are assumed to follow an exponential distribution. The probability density function (PDF) of the failure time for each unit is expressed as

$$f(x,\vartheta)=\frac{1}{\vartheta}\exp\left(-\frac{x}{\vartheta}\right), \qquad x\geq 0,\ \vartheta>0. \tag{2.1}$$

Upon subjecting the sample to a Type-II censoring scheme, the failure data is obtained in the form,$\{X_{1,n}<X_{2,n}<\cdots<X_{r,n}\}$, where $r<n$. The corresponding likelihood function is obtained as (see Khoolenjani and Shahsanaie (2016))

$$L(x,\vartheta)=\frac{n!}{(n-r)!}\frac{1}{\vartheta^r}\exp\left\{-\frac{1}{\vartheta}\sum_{i=1}^{r}x_{i,n}-\frac{(n-r)}{\vartheta}x_{r,n}\right\}, \tag{2.2}$$

where $x_{i,n}$ is the realization of the random variable $X_{i,n}$, $i=1,2,\cdots,r$.

Thus, the maximum likelihood estimator of ϑ based on the Type-II censored sample is obtained as

$$\widehat{\theta}_{MLE}=\frac{1}{r}\left\{\sum_{i=1}^{r}X_{i,n}+(n-r)X_{r,n}\right\}. \tag{2.3}$$

Here it is assumed that ϑ follows the conjugate prior probability distribution, inverse gamma distribution with scale a and shape b, and its probability density function is given by

$$\pi(\vartheta)=\frac{a^b}{\Gamma(b)}\vartheta^{-(b+1)}\exp\left(-\frac{a}{\vartheta}\right), \quad \vartheta>0,\ a,b>0. \tag{2.4}$$

The inverse gamma distribution is used as the prior distribution because it serves as the natural conjugate prior to the exponential distribution. Additionally, its flexible form, which can be adjusted by varying its parameters, allows for a reasonable representation of a particular prior knowledge.

Then the posterior density function of ϑ based on the observed data and the inverse gamma prior for ϑ given by equation (2.4) becomes

$$\begin{aligned} f(\vartheta \mid Data) &= \frac{\pi(\vartheta)L(x,\vartheta)}{\int\limits_0^\infty \pi(\vartheta)L(x,\vartheta)d\theta} \\ &= \frac{\left(r\widehat{\theta}_{mle}+a\right)^{b+r}}{\Gamma(b+r)}\vartheta^{-(b+r+1)}\exp\left\{-\frac{1}{\vartheta}\left(r\widehat{\theta}_{mle}+a\right)\right\}, \end{aligned} \tag{2.5}$$

where $\widehat{\theta}_{mle} = \left\{\sum\limits_{i=1}^{r} x_{i,n} + (n-r)x_{r,n}\right\}/r$ is the maximum likelihood estimate of ϑ. Thus, the posterior distribution of ϑ is again an inverse gamma distribution with scale parameter $\left(r\widehat{\theta}_{mle}+a\right)$ and shape $(b+r)$.

In this proposed method, Bayesian inference is conducted using the squared error loss (SEL) function, which is symmetric. Its symmetry is illustrated by the expression (see Mathai and Kumar (2024); Peng and Yan (2013))

$$l(\vartheta,\hat{\vartheta}) = (\hat{\vartheta}-\vartheta)^2,$$

where $\hat{\vartheta}$ represents the estimate of the unknown parameter ϑ.

The Bayesian estimate of any function $\delta = \delta(\vartheta)$ under the SEL is expressed as

$$\widehat{\delta}_B = E(\delta \mid \text{Data}) = \frac{\int_\vartheta \delta(\vartheta)L(x,\vartheta)\pi(\vartheta)d\vartheta}{\int_\vartheta L(x,\vartheta)\pi(\vartheta)d\vartheta}. \tag{2.6}$$

Thus, from equation (2.6), the Bayesian estimator of $\delta(\vartheta) = \vartheta$ under the SEL function is the mean of the posterior distribution and is obtained as

$$\widehat{\vartheta}_B = \frac{r\,\widehat{\theta}_{MLE}+a}{b+r-1}, \tag{2.7}$$

provided that $b+r>1$.

By using censoring schemes in deriving ASPs, we can significantly reduce both time and cost. Specifically, in this study, testing is carried out using Type-II censoring, where the test is terminated after a fixed number of failures, r, are observed. In traditional variable ASPs (Kumar and Ramyamol (2016)), the minimum value of r is determined to minimize testing costs, subject to controlling Type I and Type II errors. However, these methods often involve solving complex nonlinear equations, which can be time-intensive and computationally demanding. To address these challenges, this study proposes the use of a decision tree algorithm in variable ASP, which incorporates the costs of accepting, rejecting, or conducting 100% screening of the lot. The objective is to develop a variable ASP and determine the optimal minimum value of r to achieve a decision on lot acceptance or rejection with minimal cost. This approach provides a more efficient and cost-effective solution compared to traditional ASPs.

Here, testing begins by selecting a sample from the lot and it is terminated once a predetermined number of failures have been observed. Applying prior knowledge from previous tests, the probability density function for the lot can be obtained. The optimal value of r is determined using a backward induction approach through a decision tree model, which evaluates the probabilities associated with the optimal decision outcome. The model assumes prior information about the failure data X, the parameter ϑ, and incorporates uncertainty regarding the decision costs at the terminal nodes of the decision tree. The possible decisions concerning the lot are represented by the set $\Delta = \{D_1, D_2, D_3\}$, where D_1 signifies "accepting the lot", D_2 means "rejecting the lot", and D_3 denotes "100% testing".

The proposed ASP under the Type-II censoring scheme is outlined as follows:

1. A random sample of size n is drawn from the lot, and testing continues until r failures are observed. The resulting failure times are recorded as $\{X_{1,n} < X_{2,n} < \cdots < X_{r,n}\}$, where $r < n$ is pre-specified.
2. Using the observed data, compute the Bayesian estimate of ϑ under the SEL function, $\widehat{\vartheta}_B$, given by equation (2.7).
3. Accept the lot if $\widehat{\vartheta}_B \geq t_0$; otherwise, reject the lot.

According to the proposed ASP, the decision set for a fixed sample size n can be represented as $R = \{\gamma_r \mid r < n\}$, where γ_r denotes the decision to terminate testing after the observation of the r^{th} failure. Upon reaching a decision from set R, the corresponding failure data set of size r, $X = \{X_{1,n} < X_{2,n} < \cdots < X_{r,n}\}$, is obtained, allowing for the computation of the Bayesian estimate $\widehat{\vartheta}_B$. Furthermore, the value of the threshold t_0 must be determined within the ASP. The optimal decision depends on identifying the optimal values of (n, R, t_0), which collectively minimize the cost associated with decision-making, ensuring that the cost of the final decision is minimized.

2.1 Construction of cost functions

In this section, the following assumptions are considered for deriving the cost function corresponding to the possible decisions in Δ in the proposed ASP:

- The loss incurred due to the presence of defects per item, x, in an accepted lot is denoted by $l_0(x)$. Here we consider the quadratic loss function for $l_0(x)$, which is expressed as (see González and Palomo (2003))
$$l_0(x) = kx^2, \tag{2.8}$$
for a positive constant k.
- Based on quadratic loss function, the conditional expected loss per unit for a given ϑ is given by
$$\begin{aligned} L_0(\vartheta) &= \int_0^\infty l_0(x) f(x,\vartheta) dx \\ &= 2k\vartheta^2. \end{aligned} \tag{2.9}$$
- In this model, prior information about the failure data and the parameter, ϑ is taken into account, while the costs associated with decisions at the terminal nodes of the decision tree remain uncertain. The cost function corresponding to the decision Δ for a given set of parameters (n, R, t_0) is a random variable. Thus, the expected cost of the three decisions in Δ are derived following the methods described in González and Palomo (2003).

  (I) *Cost of accepting the lot*: The cost of accepting a lot by testing a sample of size n from a lot of size N is derived as follows:
  From the proposed ASP, the probability of accepting the lot for a given value of ϑ is obtained as
$$\begin{aligned} P(Accept \mid \vartheta) &= P\left(\widehat{\vartheta}_B \geq t_0 \mid \vartheta\right) \\ &= P\left(\frac{r\,\widehat{\theta}_{MLE} + a}{b + r - 1} \geq t_0\right). \end{aligned} \tag{2.10}$$

According to Epstein (1954), $\frac{2r\widehat{\theta}_{MLE}}{\vartheta}$ follows chi-square distribution with $2r$ degrees of freedom. Thus, equation (2.10) becomes

$$\begin{aligned} P(Accept \mid \vartheta) &= P\left[\chi^2_{(2r)} \geq \frac{2}{\vartheta}\{(b+r-1)t_0 - a\}\right] \\ &= \sum_{i=0}^{r-1} \frac{1}{i!}\left\{\frac{(b+r-1)t_0 - a}{\vartheta}\right\}^i \exp\left[-\frac{\{(b+r-1)t_0 - a\}}{\vartheta}\right]. \end{aligned} \tag{2.11}$$

Considering all the possible values of the parameter ϑ, the probability of accepting the lot becomes

$$P_A = \int_0^\infty P(Accept \mid \vartheta)\pi(\vartheta)d\vartheta. \tag{2.12}$$

Therefore, the expected cost of accepting the lot, EC_A, is obtained as

$$EC_A = N\int_0^\infty L_0(\vartheta)P(Accept \mid \vartheta)\pi(\vartheta)d\vartheta. \tag{2.13}$$

Substituting equations (2.4), (2.9) and (2.11) in equation (2.13), we get

$$EC_A = 2kN\frac{a^b}{\Gamma(b)}\{(b+r-1)t_0\}^{2-b}\sum_{i=0}^{r-1}\frac{\Gamma(b+i-2)}{i!}\left\{1 - \frac{a}{(b+r-1)t_0}\right\}^i. \tag{2.14}$$

Note that the total testing time under Type-II censoring is given by $T = \sum_{i=1}^{r} X_{i,n} + (n-r)X_{r,n}$ and from Epstein (1954), we have $\frac{2T}{\vartheta}$ follows chi-square distribution with $2r$ degrees of freedom. If C_t denotes the cost of testing a unit in the lot per unit time, then the expected cost of testing, EC_T, is obtained as

$$EC_T = C_t E(T) = C_t r\vartheta. \tag{2.15}$$

Thus, the total expected cost of accepting the lot, TC_A, is given by

$$TC_A = EC_A + EC_T. \tag{2.16}$$

(II) *Cost of rejecting the lot*: Let C_r denote the cost of rejecting a unit item. The probability of rejecting a lot is given by

$$P_R = 1 - P_A.$$

Then, from equation (2.12), we get

$$P_R = 1 - \frac{a^b}{\Gamma(b)}\{(b+r-1)t_0\}^{-b}\sum_{i=0}^{r-1}\frac{\Gamma(b+i)}{i!}\left\{1 - \frac{a}{(b+r-1)t_0}\right\}^i. \tag{2.17}$$

The expected cost of rejecting the lot, EC_R, is given by

$$EC_R = NC_r P_R, \tag{2.18}$$

and substituting equation (2.17) into (2.18) , the expression for EC_R becomes

$$EC_R = NC_r\left[1 - \frac{a^b}{\Gamma(b)}\{(b+r-1)t_0\}^{-b}\sum_{i=0}^{r-1}\frac{\Gamma(b+i)}{i!}\left\{1 - \frac{a}{(b+r-1)t_0}\right\}^i\right]. \tag{2.19}$$

From equation (2.15), the total expected cost of rejection of the lot, TC_R, is then given by

$$TC_R = EC_R + EC_T. \tag{2.20}$$

(III) *Cost of 100% testing the lot*: Let C_s denote the cost of 100% testing per unit time. The expected cost corresponding to 100% testing of lot, EC_S is obtained as

$$EC_S = C_s N\vartheta, \tag{2.21}$$

where $N\vartheta$ is the expected total testing time. Using equation (2.15), the total expected cost of 100% testing the lot, TC_S, is given by

$$TC_S = EC_S + EC_T. \tag{2.22}$$

2.2 Algorithm

The following algorithm is applied for the decision tree method in the variable ASP:

1. Using the Bayesian rule for continuous distributions, we get the posterior PDF as given in equation (2.5)

$$f(\vartheta \mid n, \gamma_r, t_0) = f(\vartheta \mid Data).$$

2. The corresponding total costs for each decision in Δ for a given set of (n, R, t_0) and data set X, is obtained as in equations (2.16), (2.20) and (2.22), that is
   (i) Cost of accepting the lot, $c(n, \gamma_r, t_0, D_1)$ = $EC_A + EC_T$.
   (ii) Cost of rejecting the lot, $c(n, \gamma_r, t_0, D_2)$ = $EC_R + EC_T$.
   (iii) Cost of testing the entire lot, $c(n, \gamma_r, t_0, D_3)$ = $EC_S + EC_T$.
3. Next, the conditional expected value of c is calculated since EC_T is in terms of the unknown quantity ϑ

$$c^*(n, \gamma_r, t_0, D_j) = \int_0^\infty c(n, \gamma_r, t_0, D_j) f(\vartheta \mid n, \gamma_r, t_0) d\vartheta. \tag{2.23}$$

4. Applying the backward induction algorithm in the decision tree, first evaluate

$$c^*(\gamma_r, t_0, D_j) = \underset{n}{\text{Min}}\ c^*(n, \gamma_r, t_0, D_j). \tag{2.24}$$

5. Then find

$$c^*(\gamma_r, D_j) = \underset{t_0}{\text{Min}}\ c^*(\gamma_r, t_0, D_j). \tag{2.25}$$

6. Next for the optimal decision, find

$$c^*(D_j) = \underset{\gamma_r}{\text{Min}}\ c^*(\gamma_r, D_j). \tag{2.26}$$

7. For the required optimal cost, evaluate

$$c^* = \underset{D_j}{\text{Min}}\ c^*(D_j), \tag{2.27}$$

such that the optimal decision cost is obtained for a $\gamma_r,\ r < n$, while satisfying the decision criteria.

The illustration of the algorithms is given using Figure 1. The plan parameters n, t_0, and γ_r are fixed step by step of the algorithm, and finally, the decision costs can be calculated to identify the optimal minimum cost and corresponding decision. The optimal decision is to select the value of r where optimal decision cost c^* is minimized, ensuring that it meets the decision criteria so that testing will stop after the r^{th} failure. Thus, optimal decision involves selecting γ_r for a combination of (n, t_0), which gives the failure data set X, and the decision regarding the lot is D_j. The decision maker relies on an inspection strategy or prior information to choose (n, t_0) and Δ, although γ_r cannot be arbitrarily chosen. The proposed model aids in determining the optimal decision based on known parameters.

3 Illustration of the proposed ASP: Case study and simulated data analysis

3.1 Case study analysis of failure time data of electronic appliances

This section illustrates the application of the proposed decision tree method in variable ASP using a real-world example from Noughabi (2015); Mathai and Kumar (2024). The failure rates for 36 compact electrical appliances that underwent an automated life test are presented in Table 1. The lifetimes refer to the number of operational cycles each appliance could endure before failure. The failure data follows an exponential distribution, making it suitable for the Bayesian decision tree approach discussed in Section 2. To validate and demonstrate

Table 1 *Failure times (cycles) of 36 appliances.*

11	35	49	170	329	381	708	958	1062	1167	1594	1925
1990	2223	2327	2400	2451	2471	2551	2565	2568	2694	2702	2761
2831	3034	3059	3112	3214	3478	3504	4329	6367	6976	7846	13403

the proposed model, let $N = 500$, $k = 1$ and consider different sets of values for $(n \leq 36,\ t_0)$ given in Table 2. In practical situations, the experimenter determines these values based on previous experience or particular criteria. Here, the Bayesian estimator is computed using the hyper-parameters $a = 1.25$ and $b = 2.5$, which provide a more accurate Bayesian estimate for ϑ (see Lin, Huang and Balakrishnan (2008); Yeh (1994)). In addition, the cost parameters are set as $C_r = 1$, $C_s = 1$, and $C_t = 1$, to evaluate the total expected cost across various decision outcomes. These cost parameters are user-defined and can be considered based on the specific context and requirements of the experiment. From equation (2.23), $c^*(n, \gamma_r, t_0, D_j)$ for each

Table 2 *Values of the decision parameters.*

n	$36, 31, 30, 29, 27$	Sample size
t_0	$2065, 2300, 2500$	Decision threshold
R	$\gamma_r = r, r < n$	Fixed number of failures to stop the test
Δ	D_1	Accept the lot
	D_2	Reject the lot
	D_3	100% testing

case of decision in Δ is obtained as

$$
\begin{aligned}
c^*(n, \gamma_r, t_0, D_1) &= EC_A + C_t r \widehat{\vartheta}_B, \\
c^*(n, \gamma_r, t_0, D_2) &= EC_R + C_t r \widehat{\vartheta}_B, \\
c^*(n, \gamma_r, t_0, D_3) &= (C_s N + C_t r) \widehat{\vartheta}_B.
\end{aligned} \tag{3.1}
$$

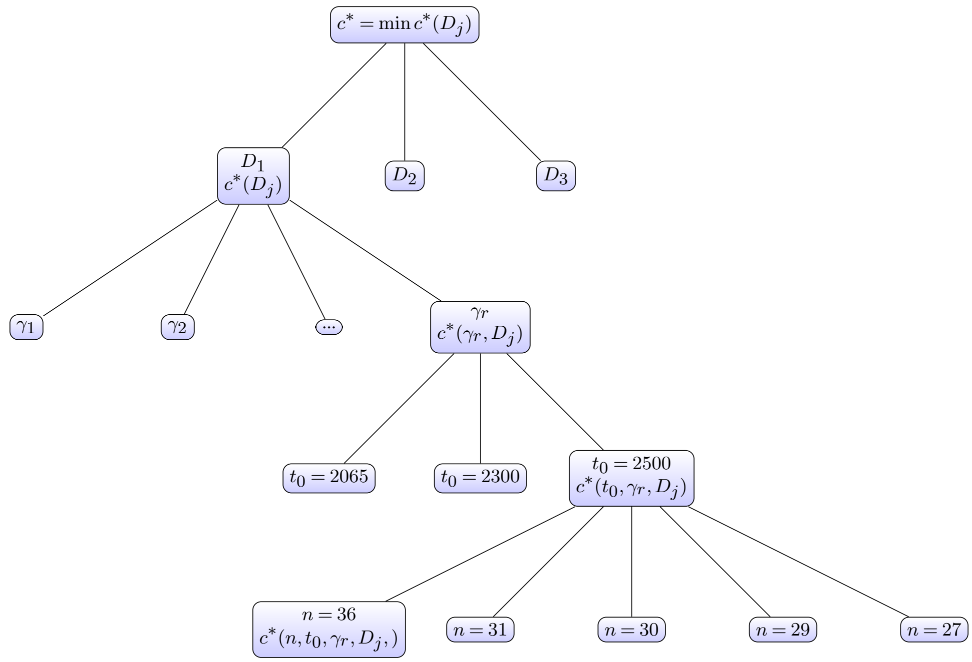

Figure 1 *Decison tree of the case study.*

Applying the algorithm explained in Subsection 2.2, we get the optimal decision corresponding to the optimal minimum decision cost aligned with the decision criteria. By implementing the backward induction algorithm from Step 4 to Step 7 for cost optimization, as illustrated in Figure 1, we compute the value of $r < n$ and corresponding minimum decision costs, satisfying the decision criteria, for each combination of fixed values of n and t_0 presented in Table 2. The value of r and the associated minimum decision cost satisfying the decision criteria are obtained and are tabulated in Table 3 .

Table 3 *Values of r and corresponding minimum decision costs for $C_r = C_s = C_t = 1$.*

n	t_0	r	$c^*(n, \gamma_r, t_0, D_1)$	$c^*(n, \gamma_r, t_0, D_2)$	$c^*(n, \gamma_r, t_0, D_3)$
36	2065	7	18343.5747	18795.7353	1325133.9706
	2300	7	18341.0655	18795.7353	1325133.9706
	2500	7	18339.215	18795.7353	1325133.9706
31	2065	7	15428.7961	15880.7961	1113983.3823
	2300	8	20826.4356	20279.9999	1319530
	2500	8	20824.54	20279.9999	1319530
30	2065	7	14845.8404	15297.3824	1071753.2647
	2300	8	20019.7978	20473.2632	1268302.2106
	2500	11	32386.9227	32841.0999	1502391.100
29	2065	8	19215.7412	19666.5263	1217074.4211
	2300	8	19213.1599	19666.5263	1217074.4211
	2500	11	30984.2997	31438.3799	1437228.38
27	2065	8	17602.4765	18053.0526	1114618.8421
	2300	11	28181.0174	28632.9399	1306902.94
	2500	11	28179.0548	28632.9399	1306902.94

The cost analysis presented in Table 3 suggests that the optimal decision is to accept the lot for $n = 30$, $t_0 = 2065$, and the corresponding value of $r = 7$ with $c^* = 14845.8404$. That is, the lot can be accepted with $14,845.8404 under Type-II censoring by terminating the test after observing 7 failures from a sample of 30.

The same dataset is used in Mathai and Kumar (2024), where they designed a variable ASP under a Type-I hybrid censoring scheme using the Bayesian estimate of ϑ. In their approach, they employed the complex conventional method of minimizing the overall expected testing cost, subject to Type I and Type II error, with a power test of 80%, to obtain the optimal values of n, t_0, and r. According to their model, the lot is accepted, but testing is terminated only after 9 failures. Additionally, they computed the cost without incorporating any loss functions. Conversely, the proposed variable ASP with the decision tree method is simpler, as it does not require complex nonlinear or stochastic optimization to determine the optimal parameter values. It also offers the advantage of achieving a smaller value of r, allowing for earlier test termination. Also, for the same user-defined values of acceptable and rejecting quality level of life, the power of the test in our approach is significantly higher, reaching about 98%. However, the inclusion of the loss function in our approach results in higher cost, as we account for the cost of defects per item in the accepted lot. This makes our model more comprehensive in terms of cost evaluation.

3.2 Illustration of ASP using simulated data

In this section, a sensitivity analysis study is conducted to analyze the impact of parameter variations on the optimal decision cost. Simulated exponential data of size 50 with a mean of 100 is used for this purpose. Initially, the proposed ASP is demonstrated using the simulated data, presented in Table 4, and then the prior and cost parameters are varied, to study how these changes affect the performance and cost-effectiveness of the proposed sampling plan. As in Subsection 3.1, consider the parameters $a = 1.25$, $b = 2.5$, $C_r = 1$, $C_s = 1$, and $C_t = 1$,

Table 4 *Failure lifetimes of 50 units with a mean life of 100.*

20.4906	9.8946	206.3672	9.0608	45.8298	232.7489	127.8344	60.3523	4.3422
184.7612	2.9848	4.3777	72.2832	22.2793	195.2729	86.3316	8.8028	23.2932
42.1990	333.2278	16.3544	6.8286	38.7524	27.7415	29.6881	93.5914	42.2391
34.8075	344.7257	128.4016	307.5395	233.1687	19.4243	36.4090	114.8540	5.1060
82.3838	96.3491	26.7204	22.9162	167.7328	71.3831	80.8365	43.6471	34.3385
3.5743	4.1351	176.5001	336.8360	28.1453				

to compute the expected decision costs given in equation 3.1. By applying the decision tree algorithm using the backward induction method outlined in Subsection 2.2, with decision parameters $n = 50, 45, 40, 35, 30, 25$, $t_0 = 150, 200, 250, 300$, and γ_r for $r < n$, the optimal decision is identified as the one that minimizes costs while meeting the decision criteria. The value of $r < n$ and the corresponding minimum decision costs are computed for each combination of fixed values of n and t_0, as shown in Figure 1. The optimal minimum decision cost and the associated value of r satisfying the decision criteria are obtained and are summarized in Table 5. The cost analysis presented in Table 5 indicates that the optimal decision is to accept the lot when $n = 35$ and $t_0 = 250$, with $r = 4$ and an associated minimum cost of $c^* = 1216.9156$. This means the lot can be accepted with an acceptance cost of 1216.9156 under Type-II censoring, where the test is terminated after observing the first 4 failures from a sample of 50.

Additionally, a sensitivity analysis study is performed using the simulated data in Table 4 to assess how variations in cost parameters, (C_t, C_r, C_s), and prior parameters, (a, b), influence the optimal decision cost and the corresponding decision on the lot. Figure 2 shows the results of varying C_t while keeping C_r, C_s fixed. It is observed that all three decision costs

Table 5 *Values of r and corresponding minimum decision costs for $C_r = C_s = C_t = 1$ with simulated data.*

n	t_0	r	$c^*(n, \gamma_r, t_0, D_1)$	$c^*(n, \gamma_r, t_0, D_2)$	$c^*(n, \gamma_r, t_0, D_3)$
50	150	6	2163.3356	2494.4583	168199.3790
	200	6	2140.778	2494.4583	168199.3790
	250	6	2125.6713	2494.4586	168199.3790
	300	6	2113.9833	2494.4588	168199.3790
45	150	5	1664.0688	1997.3	151227.4398
	200	5	1641.7879	1997.3007	151227.4398
	250	5	1626.5672	1997.3010	151227.4398
	300	4	1529.9692	1905.0416	177035.2874
40	150	6	1758.6116	2085.9808	133751.1689
	200	6	1735.5528	2085.9815	133751.1689
	250	6	1719.7994	2085.9818	133751.1689
	300	7	2251.1361	2614.9703	153814.3119
35	150	6	1556.2496	1881.7424	116527.0638
	200	6	1532.9402	1881.7431	116527.0638
	250	4	1216.9156	1589.2423	137244.5790
	300	11	4249.9464	4619.2036	191355.7479
30	150	6	1353.8876	1677.5039	99302.9588
	200	7	1708.1159	2052.6139	112453.6617
	250	11	3544.3403	3899.6107	157927.3952
	300	11	3531.7595	3899.6109	157927.3952
25	150	7	1452.8153	1771.4351	92088
	200	16	3766.0969	4098.8903	116064.2426
	250	12	3236.2637	3589.1221	131802.5671
	300	11	3093.3349	3173.4198	120003.559

Table 6 *Effect of varying cost parameters C_t, C_r and C_s in decision costs.*

C_t	C_r	C_s	$c^*(n, \gamma_r, t_0, D_1)$	$c^*(n, \gamma_r, t_0, D_2)$	$c^*(n, \gamma_r, t_0, D_3)$
50	100	150	54589.8074	104462.1019	20477762.5891
60	110	160	65482.2343	120354.5256	21850208.3796
70	120	170	76374.6612	136246.9492	23222654.1702
80	130	180	87267.0881	152139.3729	24595099.9607
90	140	190	98159.5150	168031.7966	25967545.7513
100	150	200	109051.9420	183924.2202	27339991.5418

increase monotonically with the rise in C_t, resulting in a minimum acceptance cost. This indicates that the acceptance cost increases as C_t increases. In contrast, Figure 3, where C_r is varied while the other parameters remain fixed, shows that the acceptance cost remains constant, as it does not depend on C_r. Similarly, Figure 4 shows the same pattern when C_s is varied, with the acceptance cost remaining constant. Next, the analysis considers cases where two parameters are varied while keeping the third one fixed. Figure 5 illustrates the changes in optimal acceptance cost when C_s is held constant, while C_t and C_r are varied. The results show that the acceptance cost increases as both C_t and C_r rise. Similarly, in Figure 6 where C_t and C_s are varied with C_r fixed, an increase in the acceptance cost is observed. However, when C_t is held constant and both C_r and C_s are varied, the acceptance cost remains unchanged, as illustrated in Figure 7. Furthermore, when all three cost parameters are increased simultaneously, the acceptance cost also increases, as summarized in Table 6. Additionally, Figure 8 demonstrates the effect of varying the prior parameters a and b. It shows that as a and b increases, acceptance cost decreases.

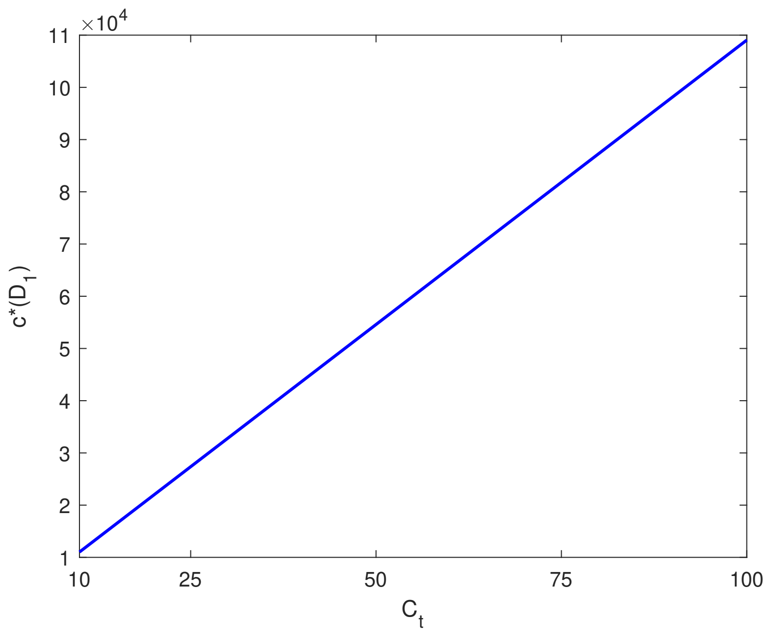

Figure 2 *Effect of varying C_t on optimal cost with $C_r = 100$, $C_s = 150$, $a = 1.25$, and $b = 2.5$.*

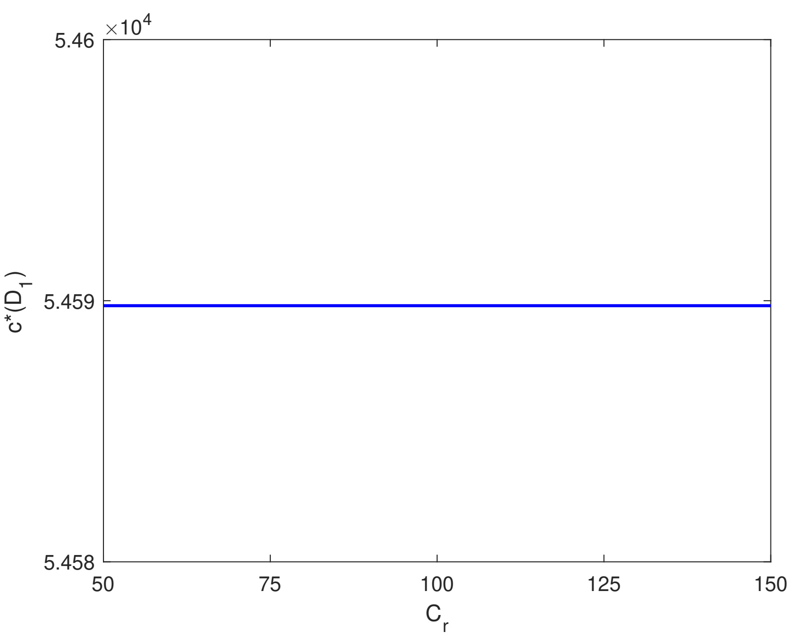

Figure 3 *Effect of varying C_r on optimal cost with $C_t = 50$, $C_s = 150$, $a = 1.25$, and $b = 2.5$.*

4 Conclusions

In real-world scenarios, conducting life tests on every item in a lot is often impractical due to time and cost constraints, especially when testing is destructive. Censoring schemes, such as Type-II censoring, are crucial in variable ASP as they allow testing to terminate after a fixed number of failures, thereby minimizing both the duration of testing and the associated

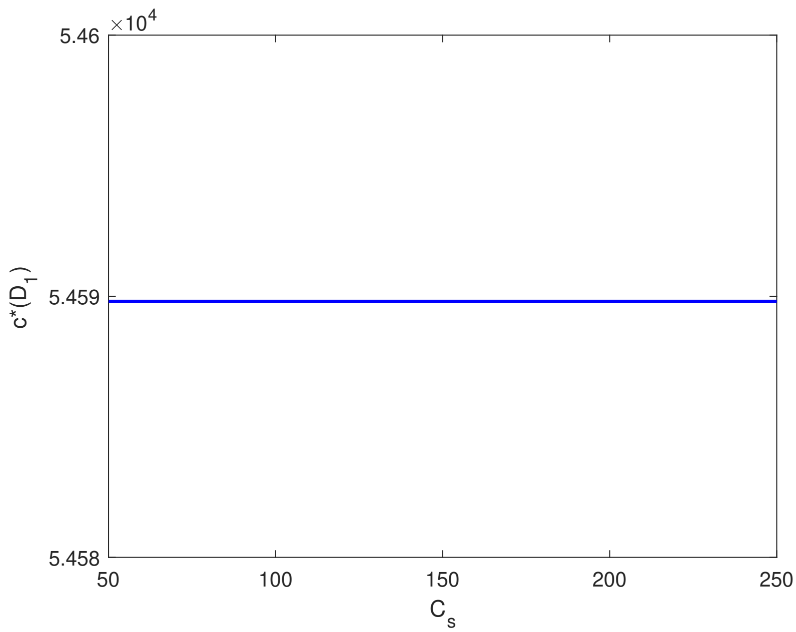

Figure 4 *Effect of varying C_s on optimal cost with $C_t = 50$, $C_r = 100$, $a = 1.25$, and $b = 2.5$.*

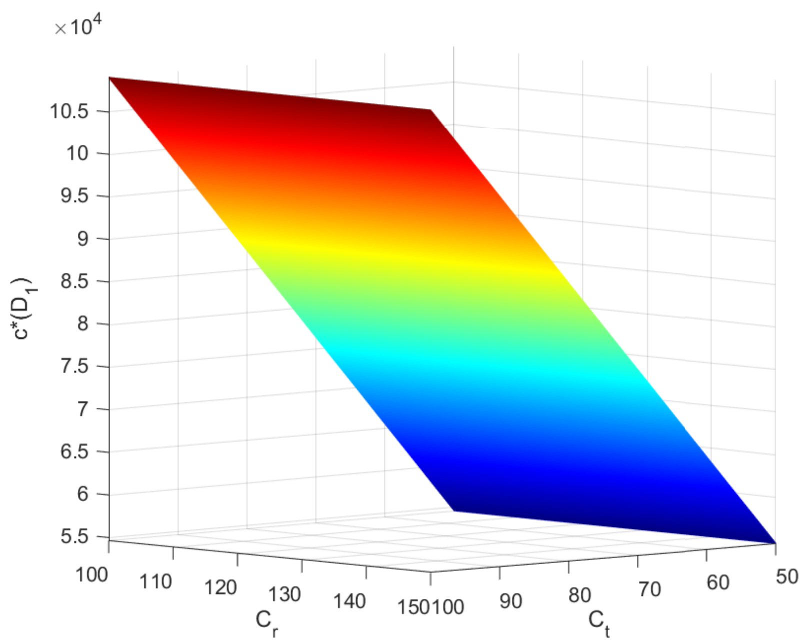

Figure 5 *Effect of varying both C_t and C_r on optimal cost with $C_s = 150$, $a = 1.25$, and $b = 2.5$.*

expenses. In this model, a variable ASP under Type-II censoring, is developed assuming an exponential distribution for failure times with an inverse gamma prior for the parameter estimation.

The proposed approach introduces a novel application of the decision tree method for variable ASP design under Type-II censoring. Unlike traditional methods, which often involve

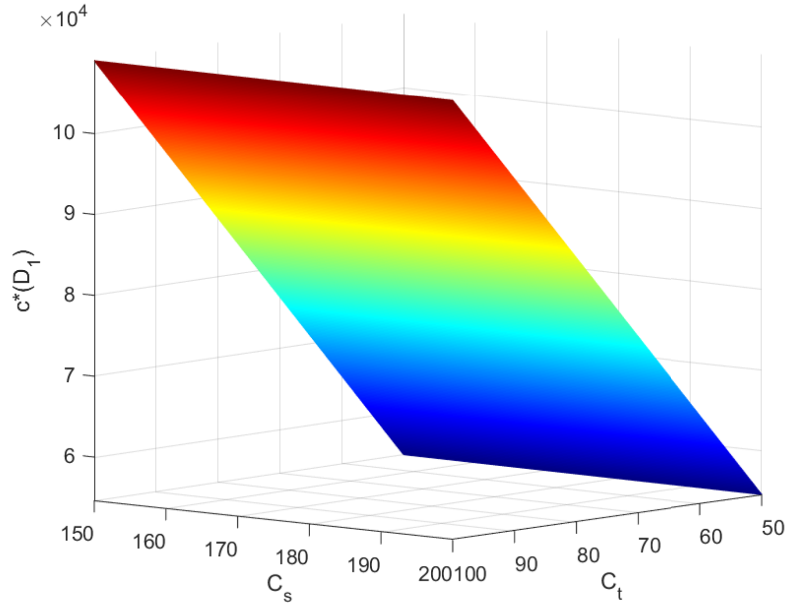

Figure 6 *Effect of varying both C_t and C_s on optimal cost with $C_r = 100$, $a = 1.25$, and $b = 2.5$.*

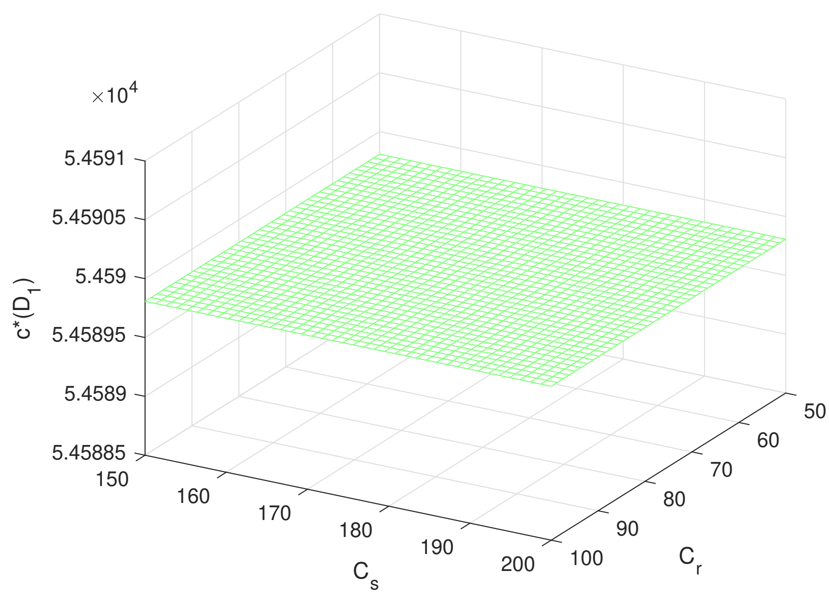

Figure 7 *Effect of varying both C_r and C_s on optimal cost with $C_t = 50$, $a = 1.25$, and $b = 2.5$.*

complex nonlinear or stochastic optimization to determine optimal parameters, this model simplifies the process by using the decision tree algorithm. This method allows for efficient decision-making and obtains the minimum number of failures (r) to terminate the test, without the need for intricate optimization techniques, making it highly practical for real-world applications.

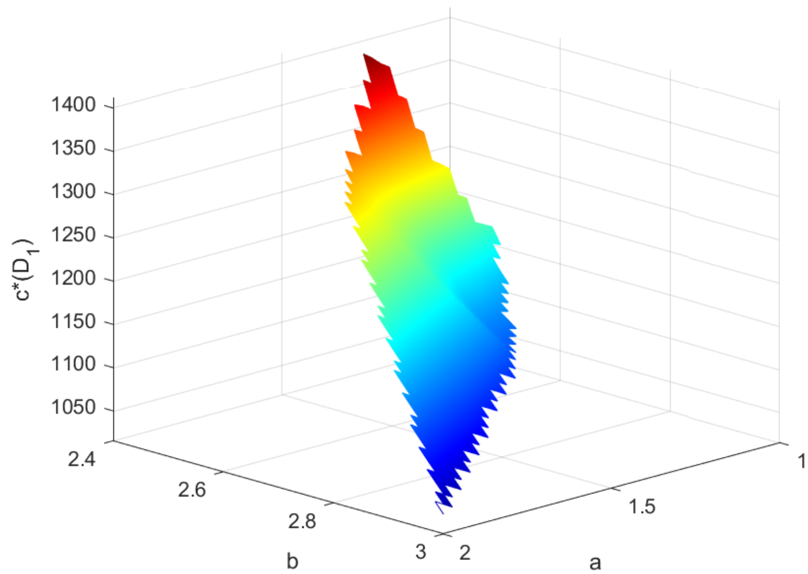

Figure 8 *Effect of varying prior parameters a and b on optimal cost with* $C_t = C_r = C_s = 1$.

By applying Bayesian inference, the model effectively incorporates prior knowledge and updates decisions as new data is observed, which reduces the need for excessive testing while ensuring product reliability. The key strength of this approach lies in its ability to provide the optimal decision, along with the minimum cost and the minimum number of failures, r, needed to terminate testing. The decision tree method ensures the computational simplicity of the proposed plan and immediate decision-making irrespective of the sample size. A real-life case study is conducted to illustrate the practical application of the proposed model. Compared to the model presented in Mathai and Kumar (2024), which employs conventional methods to design a variable ASP under Type-I hybrid censoring, our approach offers the advantage of a smaller number of failures (r) required for test termination. Furthermore, our plan has about a 24% increase in the power of test, providing a stronger capability to detect defects. While their method involves complex nonlinear optimization without considering loss functions, our decision tree-based model simplifies the process but includes loss functions, making it more comprehensive for cost evaluation. Additionally, sensitivity analysis is performed using simulated data to examine the effect of varying cost parameters and prior parameters on the optimal decision cost. To further enhance the practical applications of the decision tree algorithm in reliability testing, it is possible to incorporate progressive censoring and accelerated life testing as a future area of research.

Acknowledgments

The authors are thankful to the computational facilities extended by the DST FIST LAB (No. SR/FST/MS-I/2019/40) in the Department of Mathematics, National Institute of Technology Calicut. Authors would like to thank the Editor and the reviewers for meticulous reading and constructive comments which have greatly improved the presentation and quality of this manuscript.

References

ALSULTAN, R. and AL-OMARI, A. (2023). Zeghdoudi distribution in acceptance sampling plans based on truncated life tests with real data application. *Decision Making: Applications in Management and Engineering* **6**, 432–448. https://doi.org/10.31181/dmame05012023a

AMINZADEH, M. S. (2003). Bayesian economic variable acceptance-sampling plan using inverse Gaussian model and step-loss function. *Communications in Statistics-Theory and Methods* **32**, 961–982. https://doi.org/10.1081/STA-120019956

AZAM, M., ASLAM, M. and NIAKI, S. T. A. (2020). A repetitive sampling plan using decision trees method. *Journal of Statistics and Management Systems* **23**, 789–807. https://doi.org/10.1080/09720510.2019.1694738

BALAKRISHNAN, N. and CRAMER, E. (2014). *The Art of Progressive Censoring*. *Statistics for Industry and Technology*. New York: Birkhäuser/Springer. https://doi.org/10.1007/978-0-8176-4807-7

BALAKRISHNAN, N. and KUNDU, D. (2013). Hybrid censoring: Models, inferential results and applications. *Computational Statistics & Data Analysis* **57**, 166–209. https://doi.org/10.1016/j.csda.2012.03.025

CHAKRABARTY, J. B., ROY, S. and CHOWDHURY, S. (2023). On the economic design of optimal sampling plan under accelerated life test setting. *International Journal of Quality & Reliability Management* **40**, 1683–1705. https://doi.org/10.1108/IJQRM-07-2021-0235

CHEN, L.-S., LIANG, T. and YANG, M.-C. (2021). Designing Bayesian sampling plans for simple step-stress of accelerated life test on censored data. *Journal of Statistical Computation and Simulation* **92**, 395–415. https://doi.org/10.1080/00949655.2021.1961771

CHEN, L.-S., YANG, M.-C. and LIANG, T. (2016). Curtailed Bayesian sampling plans for exponential distributions based on Type-II censored samples. *Journal of Statistical Computation and Simulation* **87**, 1160–1178. https://doi.org/10.1080/00949655.2016.1254215

EPSTEIN, B. (1954). Truncated life tests in the exponential case. *The Annals of Mathematical Statistics* **25**, 555–564. https://doi.org/10.1080/01621459.1953.10483488

FALLAHNEZHAD, M. S. and ASLAM, M. (2013). A new economical design of acceptance sampling models using Bayesian inference. *Accreditation and Quality Assurance* **18**, 187–195. https://doi.org/10.1007/s00769-013-0984-9

FALLAHNEZHAD, M. S. and BABADI, A. Y. (2015). A new acceptance sampling plan using Bayesian approach in the presence of inspection errors. *Transactions of the Institute of Measurement and Control* **37**, 1060–1073. https://doi.org/10.1177/0142331214555395

FALLAHNEZHAD, M. S., NIAKI, S. T. A. and VAHDAT ZAD, M. A. (2012). A new acceptance sampling design using Bayesian modeling and backward induction. *International Journal of Engineering* **25**, 45–54. https://doi.org/10.5829/idosi.ije.2012.25.01c.06

GONZÁLEZ, C. and PALOMO, G. (2003). Bayesian acceptance sampling plans following economic criteria: An application to paper pulp manufacturing. *Journal of Applied Statistics* **30**, 319–333. https://doi.org/10.1080/0266476022000030093

KHOOLENJANI, N. B. and SHAHSANAIE, F. (2016). Estimating the parameter of exponential distribution under Type-II censoring from fuzzy data. *Journal of Statistical Theory and Applications* **15**, 181–195. https://doi.org/10.2991/jsta.2016.15.2.8

KUMAR, M., BAJEEL, P. N. and RAMYAMOL, P. C. (2020). Optimal design of reliability acceptance sampling plan based on data obtained from partially accelerated life test (PALT). *International Journal of Reliability, Quality and Safety Engineering* **27**, 2040002. https://doi.org/10.1142/S0218539320400021

KUMAR, M. and RAMYAMOL, P. C. (2016). Design of optimal reliability acceptance sampling plans for exponential distribution. *Economic quality control* **31**, 23–36. https://doi.org/10.1515/eqc-2015-0005

LEE, A. H., WU, C.-W., WANG, T.-C. and KUO, M.-H. (2024). Construction of acceptance sampling schemes for exponential lifetime products with progressive Type-II right censoring. *Reliability Engineering & System Safety* **243**, 109843. https://doi.org/10.1016/j.ress.2023.109843

LIANG, T. and YANG, M.-C. (2011). Optimal Bayesian sampling plans for exponential distributions based on hybrid censored samples. *Journal of Statistical Computation and Simulation* **83**, 922–940. https://doi.org/10.1080/00949655.2011.642378

LIN, C.-T., HUANG, Y.-L. and BALAKRISHNAN, N. (2008). Exact Bayesian variable sampling plans for the exponential distribution based on Type-I and Type-II hybrid censored samples. *Communications in Statistics—Simulation and Computation®* **37**, 1101–1116. https://doi.org/10.1080/03610910801923869

LOU, Y., ZHENG, H. and YANG, J. (2023). Optimal design of reliability acceptance sampling plan based on constant-stress accelerated life testing considering the uncertainty of accelerated factor. *Quality and Reliability Engineering International* **39**, 1318–1333. https://doi.org/10.1002/qre.3295

MATHAI, A. M. and KUMAR, M. (2024). Optimal variable acceptance sampling plan for exponential distribution using Bayesian estimate under Type-I hybrid censoring. *Hacettepe Journal of Mathematics and Statistics* **53**, 1178–1195. https://doi.org/10.15672/hujms.1355348

NOUGHABI, H. A. (2015). Testing exponentiality based on the likelihood ratio and power comparison. *Annals of Data Science* **2**, 195–204. https://doi.org/10.1007/s40745-015-0041-0

PENG, X.-Y. and YAN, Z.-Z. (2013). Bayesian estimation for generalized exponential distribution based on progressive Type-I interval censoring. *Acta Mathematicae Applicatae Sinica. English Series* **29**, 391–402. https://doi.org/10.1007/s10255-013-0222-6

PENG, B., YANG, K. and DONG, X. (2024). Variable selection for quantile autoregressive model: Bayesian methods versus classical methods. *Journal of Applied Statistics* **51**, 1098–1130. https://doi.org/10.1080/02664763.2023.2178642

PRAJAPAT, K., KOLEY, A., MITRA, S. and KUNDU, D. (2023). An optimal Bayesian sampling plan for two-parameter exponential distribution under Type-I hybrid censoring. *Sankhya A* **85**, 512–539. https://doi.org/10.1007/s13171-021-00263-2

PRAJAPATI, D. and KUNDU, D. (2023). Bayesian acceptance sampling plan for simple step-stress model. *Communications in Statistics-Simulation and Computation* **53**, 6411—6431. https://doi.org/10.1080/03610918.2023.2234664

RAMYAMOL, P. C. and KUMAR, M. (2020). Optimal reliability acceptance sampling plan based on accelerated life test data. *International Journal of Productivity and Quality Management* **30**, 354–370. https://doi.org/10.1504/IJPQM.2020.108377

THOMAS, J. T. and KUMAR, M. (2023). An optimal Bayesian acceptance sampling plan using decision tree method. *International Journal of Applied Management Science* **15**, 311–325. https://doi.org/10.1504/IJAMS.2023.134457

TSAI, T.-R., CHIANG, J.-Y., LIANG, T. and YANG, M.-C. (2012). Efficient Bayesian sampling plans for exponential distributions with Type-I censored samples. *Journal of Statistical Computation and Simulation* **84**, 964–981. https://doi.org/10.1080/00949655.2012.736513

YANG, K., YU, X., ZHANG, Q. and DONG, X. (2022). On MCMC sampling in self-exciting integer-valued threshold time series models. *Computational Statistics & Data Analysis* **169**, 107410. https://doi.org/10.1016/j.csda.2021.107410

YEH, L. (1990). An optimal single variable sampling plan with censoring. *Journal of the Royal Statistical Society: Series D (The Statistician)* **39**, 53–66. https://doi.org/10.2307/2348194

YEH, L. (1994). Bayesian variable sampling plans for the exponential distribution with Type-I censoring. *The Annals of Statistics* **22**, 696–711. https://doi.org/10.1214/aos/1176325491

YEH, L. and CHOY, S. T. B. (1995). Bayesian variable sampling plans for the exponential distribution with uniformly distributed random censoring. *Journal of Statistical Planning and Inference* **47**, 277–293. https://doi.org/10.1016/0378-3758(94)00136-J