CONDITIONS FOR EXISTENCE AND UNIQUENESS OF THE INVERSE FIRST-PASSAGE TIME PROBLEM APPLICABLE FOR LÉVY PROCESSES AND DIFFUSIONS

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For a real-valued stochastic process $(X_t)_{t\geq 0}$ we establish conditions under which the inverse first-passage time problem has a solution for any random variable $\xi > 0$. For Markov processes we give additional conditions under which the solutions are unique and solutions corresponding to ordered initial states fulfill a comparison principle. As examples we show that these conditions include Lévy processes with infinite activity or unbounded variation and diffusions on an interval with appropriate behavior at the boundaries. Our methods are based on the techniques used in the case of Brownian motion and rely on discrete approximations of solutions via Γ-convergence from [3] and [13] combined with stochastic ordering arguments adapted from [35].

1. Introduction. Given a random variable ξ with values in $(0,\infty)$ the inverse first-passage time problem for a stochastic process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}$ consists of the question whether there exists $b:[0,\infty]\to[-\infty,\infty]$ such that the first-passage time

$$\tau_b := \inf\{t>0 : X_t \geq b(t)\}$$

of b has the same distribution as ξ. If this question can be answered affirmatively, one naturally asks whether these solutions are unique in a reasonable sense and which properties they have. The terminology is due to the first-passage time problem, where for a stochastic process $(X_t)_{t\geq 0}$ and a function b the question is to determine properties of the distribution of τ_b. Primarily, the inverse first-passage time problem was studied for Brownian motion and revealed relations to free boundary problems, integral equations and optimal stopping problems and gave rise to applications in mathematical finance. For general processes this problem is of particular interest due to its possible relevance in applications and the new theoretical questions it gives rise to.

The inverse first-passage time problem roots back to the broader question of Shiryaev, whether there is a stopping time with respect to a Brownian motion which is exponentially distributed. This question was answered by [20] for a general stochastic process by establishing conditions under which stopping times with given distributions exist. For the inverse first-passage time problem the existence of lower-semicontinuous solutions was established in [3] for reflected Brownian motion by a discrete approximation of their epigraphs. In the case that $\xi > 0$ has no atoms, existence and uniqueness have been obtained for diffusions in [14] and [13] via a transfer into a free boundary problem. For Brownian motion uniqueness was shown for arbitrary $\xi > 0$ in [22] via discretization of a related optimal stopping problem and independently deduced in [6] in a more general setting of optimal stopping problems with distribution constraints. For reflected Brownian motion, in [35] the uniqueness was shown via a discrete approximation argument paired with stochastic ordering. The discretizations of

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[22] and [35] are related to the approximation in [3]. Conditions for continuity of solutions were given in [13], [22] and [40], where higher order regularity was studied in [12]. The inverse first-passage time problem and the first-passage time problem for Brownian motion are related to certain integral equations, see [39], [14], [22], [29]. Numerical approaches for the case of Brownian motion are to be found in [47], [1], [45], [26], [34] and for an Ornstein-Uhlenbeck process in [15]. Applications have been proposed in [27], [4], [41], [42], [19] in the context of modeling default, neuronal activity or failure. For Brownian motion the modification of the problem to fix both b and ξ and to ask whether X_0 can be randomly distributed such that τ_b has distribution according to ξ has been studied in [28], [30], [2], [31] and is naturally related to our comparison principle. Moreover, the inverse first-passage time problem for the case of Brownian motion and exponentially distributed ξ is related to [18], [9], [38], [8] and [7], where hydrodynamic limits of certain particle systems and corresponding free boundary problems are studied. For general ξ a related particle system whose hydrodynamic limit is characterized by the inverse first-passage time problem for reflected Brownian motion has been constructed in [34]. Another so-called soft-killing variant of the problem asks the same question but with a smoothed-out version of τ_b, where one additionally waits for an exponential clock to ring after passing the boundary, and was treated in [23], [24] and [36]. For a more detailed overview of related work in the case of Brownian motion we refer to [33]. Another modification of the problem is to fix b and the distributions of ξ and X_0 and to search for a suitable stochastic process $(X_t)_{t\geq 0}$ in order to achieve that τ_b has the same distribution as ξ. This has been studied in [16] for a certain family of Itô diffusions and in [17] for a family of processes obtained from deterministically time-changing a fixed Lévy process.

Let us summarize the methods used in this present paper. For the existence of solutions we pick up the idea of [3] from the case of reflected Brownian motion. A main ingredient in the proof of [3] is the continuity of the paths. By a careful adaption of the proof and by utilizing facts about left and right discontinuity of arbitrary functions we are able to work with quasi-left-continuous càdlàg paths instead. For the uniqueness of solutions the Markov property allows to work with stochastic orders as in [35]. In this situation we present a new and elementary argument how to infer uniqueness from the discretization of [3]. Instead of using the Wasserstein distance for the marginal distribution as in [35] we use an adapted approximation of lower semicontinuous functions from [13] and [22].

We want to emphasize the following relations to our results.

- For Brownian motion relations to free boundary problems [13], [9], optimal stopping problems [22], [6], integral equations [29], [39] and particle systems [18], [34] are known. Hence the question arises whether such relations extend to other stochastic processes.
- By the comparison principle, the inverse first-passage time problem is related to the first-passage time problem and the modified problems studied in [28] or [17], which therefore gain interest in this general setting.
- Continuity of the paths was also a main ingredient in the soft-killing inverse first-passage time problem for Brownian motion in [24], [36]. Thus it is natural to ask if this can also be extended to processes with discontinuous paths.
- Our results give more credibility to numerical approaches where existence and uniqueness were assumed for the underlying processes, see [47], [15].
- The existence result allows more flexibility in applications, since the process used for modeling purposes as in [27], [41], [19] can be chosen from a broader range.
- The discrete approximation of our approach yields a possible numerical Monte-Carlo type approximation similar as in [34] for Brownian motion.
- The recent contribution [11] studies the empirical measure of the so-called N-branching Markov process. Their results rely on the assumption that, for the Markov process X under study, a solution exists for exponentially distributed ξ.

The paper is organized as follows. In Section 2 we present our main results regarding existence, uniqueness and comparison principle as well as the conditions for Lévy processes and diffusions. The proofs of the main results are to be found in Section 3, Section 4 and Section 5. The proofs regarding the conditions for Lévy processes as well as for diffusions are contained in Section 6 and Section 7.

2. Main results.

DEFINITION 2.1. We call a lower semicontinuous function $b : [0,\infty] \to [-\infty,\infty]$ *boundary function*.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ fulfilling the usual conditions. Let $(X_t)_{t\geq 0}$ be an adapted stochastic process with values in $\mathbb{R}$. For a boundary function b define, in addition to τ_b, the first-passage time variant

$$\tau_b' := \inf\{t > 0 : X_t > b(t)\}.$$

DEFINITION 2.2. We say that $(X_t)_{t\geq 0}$ is *quasi-left-continuous*, if for any non-decreasing sequence of stopping times $T_1 \leq T_2 \leq \dots$ and $T := \lim_{n\to\infty} T_n$ it holds that

$$\lim_{n\to\infty} X_{T_n} = X_T$$

almost surely on $\{T < \infty\}$.

For the existence of solutions the following assumptions are put in force:

(E1) For every $t > 0$ the probability measure $\mathbb{P}(X_t \in \cdot\,)$ is diffuse.
(E2) $(X_t)_{t\geq 0}$ has $\mathbb{P}$-a.s. right-continuous paths and is quasi-left-continuous.
(E3) For any boundary function b it holds $\mathbb{P}$-a.s. that $\tau_b = \tau_b'$.

THEOREM 2.3 (Existence). *Assume that (E1), (E2) and (E3) are fulfilled. Then, given a random variable $\xi > 0$, there exists a boundary function b such that*

$$\tau_b \overset{d}{=} \xi.$$

REMARK 2.4. Let us emphasize that the conditions for existence are not very restrictive. Condition (E1) is necessary to have solutions for any $\xi > 0$. Furthermore, although conditions (E2) and (E3) are primarily employed for technical reasons, they turn out to be very natural. For example, (E2) includes the class of Hunt processes. A condition as (E3) is needed to exclude behavior as in Example 2.6.

REMARK 2.5. If the process takes values in an interval $(-\infty, R]$ with $R < \infty$, then the assumption of (E3) implies, by taking $b \equiv R$, that $\mathbb{P}(\tau_R < \infty) = 0$.

EXAMPLE 2.6. Let $X_t = -|B_t + 1|$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Then τ_0 has the Lévy distribution with scale 1, in particular it is supported on $(0,\infty)$. On the other hand, for every $\xi > 0$ which is supported on $(0,\infty)$ a solution b must have values in $(-\infty, 0]$. This means that such a $\xi > 0$ whose distribution is strictly larger in the usual stochastic order than the distribution of τ_0 (Definition 2.7) cannot be realized as first-passage time.

DEFINITION 2.7. For two probability measures μ,ν on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, we say μ is *smaller in the usual stochastic order*, write $\mu \preceq_{\text{st}} \nu$, if and only if

$$\mu((-\infty,c]) \geq \nu((-\infty,c]) \qquad \forall c \in \mathbb{R}.$$

For a measure μ on $\mathbb{R}$ we define its support as

$$\operatorname{supp}(\mu) := \{x \in \mathbb{R} : \mu(U) > 0 \text{ whenever } U \subseteq \mathbb{R} \text{ is open and } x \in U\}.$$

For a random variable $\xi > 0$ we define

$$t^\xi := \sup \operatorname{supp}(\mathbb{P}(\xi \in \cdot\,)) = \sup\{t > 0 : \mathbb{P}(\xi > t) > 0\}.$$

For a process taking values in an interval $E \subseteq \mathbb{R}$ the following assumptions are put in force for uniqueness:

(U1) There is a family of probability measures $(\mathbb{P}_x)_{x\in E}$ such that

$$\mathbb{P}_x(X_t \in E\ \forall t \geq 0) = 1 \quad \forall x \in E$$

and $((X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ is a Markov process as per [21, Vol.1,p.77].

(U2) For probability measures μ_1, μ_2 on E and all $t > 0$, we have that $\mu_1 \preceq_{\text{st}} \mu_2$ implies

$$\mathbb{P}_{\mu_1}(X_t \in \cdot\,) \preceq_{\text{st}} \mathbb{P}_{\mu_2}(X_t \in \cdot\,).$$

(U3) For a random variable $\xi > 0$ there is $I^\xi \subseteq (0, t^\xi)$ such that for any boundary function b with values in $\overline{E}$ and $\tau_b \overset{\text{d}}{=} \xi$ we have

$$b(t) = \sup \operatorname{supp}(\mathbb{P}(X_t \in \cdot \,|\tau_b > t))$$

for all $t \in I^\xi$.

For a probability measure μ on E we define $\mathbb{P}_\mu := \int_E \mathbb{P}_x \mu(\mathrm{d}x)$.

THEOREM 2.8 (Uniqueness). *Let $E \subseteq \mathbb{R}$ be an interval and denote $\overline{E} = [L, R]$ with $L, R \in [-\infty, \infty]$. Fix a probability measure μ on E. Let $\xi > 0$ be a random variable. Assume that (U1), (U2) are fulfilled and (E1), (E2), (E3) and (U3) with $\mathbb{P} := \mathbb{P}_\mu$ and $I^\xi \subseteq (0, t^\xi)$ are fulfilled.*
Then all boundary functions b with values in $\overline{E}$ and $\tau_b \overset{d}{=} \xi$ under $\mathbb{P}$ coincide on I^ξ.

REMARK 2.9. Let us comment on the conditions for uniqueness. (U1) and (U2) are contingent on our method of proof, for which we do not anticipate problems when working with inhomogeneous Markov processes instead. However, for simplicity we refrain from doing this. (U3) is necessary for uniqueness, since otherwise we could alter values of a solution b on the set I^ξ without affecting the distribution of the first-passage time τ_b.

DEFINITION 2.10. For two random variables $\xi, \zeta > 0$ we say ξ *is smaller in the hazard rate order than* ζ, write $\xi \preceq_{\text{hr}} \zeta$, if

$$[0, t^\zeta) \to [0, 1],\ t \mapsto \frac{\mathbb{P}(\xi > t)}{\mathbb{P}(\zeta > t)}$$

is a non-increasing function.

THEOREM 2.11 (Comparison principle). *Let $E \subseteq \mathbb{R}$ be an interval. Fix two probability measures μ_1, μ_2 on E such that $\mu_1 \preceq_{\text{st}} \mu_2$. Let $\xi_1, \xi_2 > 0$ be random variables such that $\xi_1 \preceq_{\text{hr}} \xi_2$. Assume that (U1), (U2) are fulfilled and (E1), (E2), (E3) with $\mathbb{P} := \mathbb{P}_{\mu_i}$ are fulfilled. Then for $i \in \{1, 2\}$ there exist boundary functions b_i with $\tau_{b_i} \overset{d}{=} \xi_i$ under $\mathbb{P}_{\mu_i}$ such that*

$$b_1 \leq b_2$$

pointwise.

REMARK 2.12. Since in Theorem 2.11 we merely state the existence of ordered solutions, we can spare (U3) in our list of assumptions.

2.1. *Lévy processes.* In Section 6 we establish conditions for Lévy processes under which we can apply Theorem 2.3 and Theorem 2.8. We will summarize these conditions below in Theorem 2.13.

We say a Lévy process has characteristic a triple (a,σ^2,Π) if

$$-\log\left(\mathbb{E}\left[\exp(i\theta X_1)\right]\right) = i\theta a + \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x 1_{(-1,1)}(x))\Pi(\mathrm{d}x), \tag{1}$$

where $a\in\mathbb{R}$, $\sigma^2\geq 0$ and Π is a measure on $\mathbb{R}\setminus\{0\}$ with $\int_{\mathbb{R}}(1\wedge x^2)\Pi(\mathrm{d}x)<\infty$. If $(X_t)_{t\geq 0}$ is a Lévy process with $\mathbb{P}(X_0=0)=1$, then for $x\in\mathbb{R}$ let $\mathbb{P}_x$ be a measure such that $\mathbb{P}_x\left((X_t)_{t\geq 0}\in\cdot\,\right) := \mathbb{P}\left((X_t+x)_{t\geq 0}\in\cdot\,\right)$. For the following statement note that $\mathbb{P}_0\left(X_1\in\cdot\,\right)$ is diffuse if and only if we have that $\sigma^2>0$ or $\Pi(\mathbb{R})=\infty$, see for instance Theorem 27.4 in [43]. Equivalently, we could say the Lévy process is not a compound Poisson process with or without drift. On the other hand, if a probability measure μ on $\mathbb{R}$ is diffuse, then $\mathbb{P}_\mu\left(X_t\in\cdot\,\right)$ is diffuse.

THEOREM 2.13 (Lévy processes). *Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic triple (a,σ^2,Π), $\xi>0$ be a random variable and μ be a probability measure on $\mathbb{R}$. Then (U1) and (U2) are fulfilled with $E=\mathbb{R}$ and (E2) is fulfilled with $\mathbb{P}:=\mathbb{P}_\mu$.*
Existence: *We have that (E1) implies (E3). In particular, assuming (E1), there exists a boundary function b such that $\tau_b\overset{d}{=}\xi$ under $\mathbb{P}$.*
Uniqueness:

(a) *Let one of the following be fulfilled:*
  (a.i) *$(X_t)_{t\geq 0}$ has unbounded variation, i.e. $\sigma^2>0$ or $\int_{\mathbb{R}}(1\wedge|x|)\Pi(\mathrm{d}x)=\infty$,*
  (a.ii) *$0\in\operatorname{supp}(\Pi)$ and $\Pi((0,\infty))>0$.*
  Then (U3) is fulfilled with $I^\xi := (0,t^\xi)$.
(b) *Let $0\in\operatorname{supp}(\Pi)$ and $\Pi((-\infty,0))>0$.*
  Then (U3) is fulfilled with $I^\xi := \operatorname{supp}(\mathbb{P}(\xi\in\cdot\,))\cap(0,t^\xi)$.

In particular, assuming (E1) and ((a) or (b)), all boundary functions b with $\tau_b\overset{d}{=}\xi$ under $\mathbb{P}$ coincide on I^ξ.

REMARK 2.14. In order to demonstrate the phrasing of Theorem 2.13 let us mention some examples of Lévy processes:

- If the Lévy measure has infinite activity, i.e. $\Pi(\mathbb{R})=\infty$, then we have $0\in\operatorname{supp}(\Pi)$ and (E1), and thus we have existence and uniqueness.
- If $(X_t)_{t\geq 0}$ has a Brownian component we have existence and uniqueness.
- If the law of X_0 is diffuse and $(X_t)_{t\geq 0}$ is a Poisson process with jumps of constant height we have existence.
- If $(-X_t)_{t\geq 0}$ is a Gamma process and $\xi\sim\text{Exp}$, we have existence and uniqueness of solutions on $(0,\infty)$.

2.2. *Diffusions on an interval.* In Section 7 we establish conditions for diffusions under which we can apply Theorem 2.3 and Theorem 2.8. We will summarize these conditions below in Theorem 2.16.

For the definition of a diffusion on an interval we adapt Definition 5.20 of Chapter 5 of [32].

DEFINITION 2.15. Let $E\subseteq\mathbb{R}$ be an interval and $\overline{E}=[L,R]\subseteq[-\infty,\infty]$. Furthermore, let $\beta: E\to\mathbb{R}$ and $\sigma: E\to\mathbb{R}$ be Borel-measurable functions. Let $(\mathbb{P}_x)_{x\in E}$ be a family of probability measures and $(X_t)_{t\geq 0}$, $(B_t)_{t\geq 0}$ stochastic processes such that

(i) $((X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ is a strong Markov process on E,
(ii) $(X_t)_{t\geq 0}$ has continuous paths $\mathbb{P}_x$-a.s. and $(B_t)_{t\geq 0}$ is a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{P}_x$ for every $x \in E$,
(iii) with strictly monotone sequences $(\ell_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ satisfying $L < \ell_n < r_n < R$, $\lim_{n\to\infty} \ell_n = L$ and $\lim_{n\to\infty} r_n = R$ and

$$S_n := \inf\{t \geq 0 : X_t \notin (\ell_n, r_n)\}$$

it holds for all $t \geq 0$ that

$$\mathbb{P}_x\left(\int_0^{t\wedge S_n} |\beta(X_s)| + \sigma^2(X_s)\,\mathrm{d}s < \infty\right) = 1$$

and

$$\mathbb{P}_x\left(X_{t\wedge S_n} = x + \int_0^{t\wedge S_n} \beta(X_s)\,\mathrm{d}s + \int_0^{t\wedge S_n} \sigma(X_s)\,\mathrm{d}B_s \;\forall t \geq 0\right) = 1 \tag{2}$$

for all $n \in \mathbb{N}$ and all $x \in (L, R)$.

We call $(X_t)_{t\geq 0}$ a diffusion on E with coefficients β and σ.

For the following statement, note that if $\mathbb{P}(X_t \in \cdot\,)$ is diffuse, the process can hit the lower boundary L in finite time, but it cannot get stuck there.

THEOREM 2.16. *Let $(X_t)_{t\geq 0}$ be a diffusion on E with coefficients β and σ such that $\sigma \in C^1((L,R))$, $\sigma > 0$ and β is locally bounded on (L,R). Let $\xi > 0$ be a random variable and μ be a probability measure on E. If $R \notin E$, then (U1), (U2) are fulfilled and (E2), (E3) are fulfilled with $\mathbb{P} := \mathbb{P}_\mu$. If additionally (E1) holds then we have (U3) with $I^\xi = (0, t^\xi)$.* ***Existence and uniqueness:*** *In particular, assuming $R \notin E$ and (E1), there exists a unique boundary function b on $(0, t^\xi)$ with values in $\overline{E}$ such that $\tau_b \overset{d}{=} \xi$.*

REMARK 2.17. In order to demonstrate the phrasing of Theorem 2.16 and the usage of the notion of diffusion on an interval, let us mention the following example. If $(X_t)_{t\geq 0}$ is a Bessel process of dimension $\delta > 0$ on $E = [0, \infty)$ (cf. [37, p.29]), for $X_0 > 0$ and $t < S = \inf\{s \geq 0 : X_s = 0\}$ the process fulfills the stochastic integral equation

$$X_t = X_0 + \int_0^t \frac{\delta - 1}{2X_s}\,\mathrm{d}s + B_t.$$

The coefficients fulfill the conditions of Theorem 2.16 on $(0, \infty)$. Note that we have $R = \infty$. Since $\delta > 0$ the law of X_t is absolutely continuous w.r.t. Lebesgue measure and thus (E1) is fulfilled. This means that we have existence and uniqueness of solutions for any $\xi > 0$.

3. Existence: Proof of Theorem 2.3. Let us explain the role of the conditions in the proof of Theorem 2.3, which will give us a common thread. Condition (E1) is necessary to have solutions for any $\xi > 0$ and allows for the construction of a discrete approximation. For this approximation we will use the following notion of Γ-convergence. The conditions (E2) and (E3) ensure that this approximation provides a solution. The idea of this proof follows the approach of [3].

DEFINITION 3.1. We call a sequence $(b_n)_{n\in\mathbb{N}}$ of boundary functions Γ-convergent to a boundary function b, write $b_n \xrightarrow{\Gamma} b$, if and only if

(i) for every convergent sequence $(t_n)_{n\in\mathbb{N}} \subset [0,\infty]$ with $\lim_{n\to\infty} t_n = t$ holds
$$\liminf_{n\to\infty} b_n(t_n) \geq b(t),$$
(ii) for any $t \in [0,\infty]$ exists a convergent sequence $(t_n)_{n\in\mathbb{N}} \subset [0,\infty]$ with $\lim_{n\to\infty} t_n = t$ such that
$$\lim_{n\to\infty} b_n(t_n) = b(t).$$

For a boundary function b and $s \geq 0$ we define
$$b|_s(t) := \infty \mathbb{1}_{[0,s)}(t) + b\mathbb{1}_{[s,\infty]}(t).$$

PROPOSITION 3.2. *Assume that $(X_t)_{t\geq 0}$ has a.s. right-continuous paths and is quasi-left-continuous, i.e. (E2) holds. Let b be a boundary function and assume $\tau_{b|_s} \overset{d}{=} \tau'_{b|_s}$ for every $s > 0$. Further let $b_n \overset{\Gamma}{\to} b$ and assume that*
$$\lim_{s\searrow 0} \limsup_{n\to\infty} \mathbb{P}(\tau_{b_n} \leq s) = 0.$$
Then
$$\tau_{b_n} \overset{\mathbb{P}}{\to} \tau_b$$
in probability as $n \to \infty$.

Proposition 3.2 will be proved after a sequence of preliminary lemmas.

REMARK 3.3. Note that, if $\xi > 0$ is a random variable and $\tau_{b_n} \to \xi$ in distribution, then it follows by Portmanteau's theorem that
$$\lim_{s\searrow 0} \limsup_{n\to\infty} \mathbb{P}(\tau_{b_n} \leq s) \leq \lim_{s\searrow 0} \mathbb{P}(\xi \leq s) = 0.$$

REMARK 3.4. For a boundary function b we can rewrite τ_b as
$$\tau_b = \inf\{t > 0 : (t, X_t) \in \operatorname{epi}(b)\},$$
where
$$\operatorname{epi}(b) := \{(t,x) \in [0,\infty] \times [-\infty,\infty] : x \geq b(t)\}.$$
Since b is lower semicontinuous $\operatorname{epi}(b)$ is closed in $[0,\infty] \times [-\infty,\infty]$. If $(X_t)_{t\geq 0}$ has right-continuous paths so has $(t, X_t)_{t\geq 0}$, and thus τ_b as a hitting time of a closed set is a stopping time, for instance see [5].

LEMMA 3.5. *Assume that $(X_t)_{t\geq 0}$ has right-continuous paths and b is a boundary function. Then it holds*
$$X_{\tau_b} \geq b(\tau_b)$$
almost surely.

PROOF. From the definition of the first-passage time we can find a (possibly random) sequence $s_n \searrow \tau_b$, such that $X_{s_n} \geq b(s_n)$ for all $n \in \mathbb{N}$. By the right-continuity it follows that
$$X_{\tau_b} = \lim_{n\to\infty} X_{s_n} \geq \liminf_{n\to\infty} b(s_n) \geq b(\tau_b),$$
where the last inequality follows from the lower semicontinuity. □

For a boundary function b define

$$\overline{\tau}_b := \inf\{t \geq 0 : X_t \geq b(t)\}.$$

LEMMA 3.6. *Assume that $(X_t)_{t\geq 0}$ has right-continuous paths and is quasi-left-continuous. Furthermore, let $b_n \xrightarrow{\Gamma} b$. Then on $\{\liminf_{n\to\infty} \tau_{b_n} > 0\} \cup \{\tau_b = \overline{\tau}_b\}$ we have*

$$\tau_b \leq \liminf_{n\to\infty} \tau_{b_n}$$

almost surely.

PROOF. We assume that $T := \liminf_{n\to\infty} \tau_{b_n} < \infty$. Set $T_m := \inf_{n\geq m} \tau_{b_n}$ and fix $m \in \mathbb{N}$. There is a sequence $(n_k)_{k\geq 1} \subseteq \{n \in \mathbb{N} : n \geq m\}$ (possibly random) such that $\lim_{k\to\infty} \tau_{b_{n_k}} = T_m$ and $\tau_{b_{n_k}} \geq \tau_{b_{n_{k+1}}}$ for all $k \in \mathbb{N}$. If $(n_k)_{k\in\mathbb{N}}$ is bounded, we can assume without loss of generality that $(n_k)_{k\geq 1}$ is a constant sequence and set $n^{(m)} := n_1$. Then, from Lemma 3.5, we have $X_{T_m} \geq b_{n^{(m)}}(T_m)$ almost surely. On the other hand, if $n^{(m)} := \lim_{k\to\infty} n_k = \infty$, due to the Γ-convergence and the right-continuity of the paths we have

$$b(T_m) \leq \liminf_{k\to\infty} b_{n_k}(\tau_{b_{n_k}}) \leq \liminf_{k\to\infty} X_{\tau_{b_{n_k}}} = X_{T_m}$$

almost surely. Note that the (possibly random) sequence of boundary functions

$$\tilde{b}_m := \begin{cases} b_{n^{(m)}} & : n^{(m)} < \infty, \\ b & : n^{(m)} = \infty, \end{cases}$$

Γ-converges to b as $m \to \infty$ since $n^{(m)} \geq m$. Now, since $T = \lim_{m\to\infty} T_m$, we have, by the Γ-convergence and the quasi-left-continuity, that

$$b(T) \leq \liminf_{m\to\infty} \tilde{b}_m(T_m) \leq \liminf_{m\to\infty} X_{T_m} = X_T$$

almost surely. If $T > 0$, it follows directly that $\tau_b \leq T = \liminf_{n\to\infty} \tau_{b_n}$. Generally, it follows $\overline{\tau}_b \leq T$, which concludes the proof. □

LEMMA 3.7. *Let $b_n \xrightarrow{\Gamma} b$. Let $t \in (0, \infty)$ and assume that*

$$\liminf_{s\searrow t} b(s) = b(t).$$

Then there exists a sequence $t_n \to t$ with $t_n > t$ such that

$$b_n(t_n) \to b(t).$$

PROOF. Since $\liminf_{s\searrow t} b(s) = b(t)$ there is a sequence $r_m \to t$ with $r_m > t$ such that $b(r_m) \to b(t)$ as $m \to \infty$. Since $b_n \xrightarrow{\Gamma} b$ for every $m \in \mathbb{N}$ there is a sequence $r_n^m \to r_m$ such that $b_n(r_n^m) \to b(r_m)$. Without loss of generality we can assume that $r_n^m > t$. We now define two sequences $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ by a recursive scheme. For $k \in \mathbb{N}$ assume that $m_1, \dots, m_{k-1}$ and $n_1, \dots, n_{k-1}$ are already defined. Then let $m_k > m_{k-1}$ be large enough such that

$$\max(r_m - t, |b(r_m) - b(t)|) \leq \frac{1}{k} \qquad \forall m \geq m_k.$$

Further, let $n_k > n_{k-1}$ be large enough such that

$$\max(|r_n^{m_k} - r_{m_k}|, |b_n(r_n^{m_k}) - b(r_{m_k})|) \leq \frac{1}{k} \qquad \forall n \geq n_k.$$

Now define for $n \in \mathbb{N}$ the sequence

$$t_n := \sum_{k=1}^{\infty} r_n^{m_k} \mathbb{1}_{\{n_k,\dots,n_{k+1}-1\}}(n).$$

Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $\frac{2}{k} < \varepsilon$. Let $n \in \mathbb{N}$. Then, if $n \in \{n_{\tilde{k}},\dots,n_{\tilde{k}+1}-1\}$ for $\tilde{k} \geq k$, we have

$$|t_n - t| \leq |r_n^{m_{\tilde{k}}} - r_{m_{\tilde{k}}}| + |r_{m_{\tilde{k}}} - t| \leq \frac{1}{\tilde{k}} + \frac{1}{\tilde{k}} \leq \frac{2}{k} < \varepsilon.$$

and

$$|b_n(t_n) - b(t)| \leq |b_n(r_n^{m_{\tilde{k}}}) - b(r_{m_{\tilde{k}}})| + |b(r_{m_{\tilde{k}}}) - b(t)| \leq \frac{1}{\tilde{k}} + \frac{1}{\tilde{k}} \leq \frac{2}{k} < \varepsilon.$$

This shows that eventually that $t_n \to t$ with $t_n > t$ and

$$b_n(t_n) \to b(t)$$

as $n \to \infty$. □

For a boundary function b and $\varepsilon > 0$ we interpret $b + \varepsilon$ as the boundary function given by $(b+\varepsilon)(t) = b(t) + \varepsilon$.

LEMMA 3.8. *Assume that $(X_t)_{t\geq 0}$ has right-continuous paths and is quasi-left-continuous. Furthermore, let $b_n \xrightarrow{\Gamma} b$ and $\varepsilon > 0$. Then*

$$\limsup_{n\to\infty} \tau_{b_n} \leq \tau_{b+\varepsilon}$$

almost surely.

PROOF. According to Lemma A.4 the set

$$S_b := \left\{ t \in [0,\infty) : \liminf_{s \searrow t} b(s) > b(t) \right\}$$

is countable. By setting $b(0) := \liminf_{s\searrow 0} b(s)$ (which does not affect $\tau_{b+\varepsilon}$) we can assume that $0 \notin S_b$. Since b is lower semicontinuous this means that for every $t \in [0,\infty) \setminus S_b$ we have $\liminf_{s\searrow t} b(s) = b(t)$. Since $(X_t)_{t\geq 0}$ is quasi-left-continuous and has right-continuous paths we have that

$$X_{t-} = X_t \qquad \forall t \in S_b \tag{3}$$

almost surely. Assume without loss of generality that $\tau_{b+\varepsilon} < \infty$. By Lemma 3.5 we have $X_{\tau_{b+\varepsilon}} \geq b(\tau_{b+\varepsilon}) + \varepsilon$. Due to the Γ-convergence we can choose a converging sequence $t_n \to \tau_{b+\varepsilon}$ (possibly random) such that $b_n(t_n) \to b(\tau_{b+\varepsilon})$ as $n \to \infty$. We distinguish two cases.

If $\tau_{b+\varepsilon} \in S_b$, by (3), we can assume that $X_{\tau_{b+\varepsilon}-} = X_{\tau_{b+\varepsilon}}$. We have therefore

$$\lim_{n\to\infty} X_{t_n} = X_{\tau_{b+\varepsilon}}.$$

If $\tau_{b+\varepsilon} \in [0,\infty) \setminus S_b$ due to Lemma 3.7, we can assume that $t_n > \tau_{b+\varepsilon}$ for all $n \in \mathbb{N}$. Thus, since $(X_t)_{t\geq 0}$ has right-continuous paths we have

$$\lim_{n\to\infty} X_{t_n} = X_{\tau_{b+\varepsilon}}.$$

Let $N \in \mathbb{N}$ (possibly random) be large enough such that for every $n \geq N$ we have

$$b_n(t_n) \leq b(\tau_{b+\varepsilon}) + \frac{\varepsilon}{2} \quad \text{and} \quad X_{t_n} \geq b(\tau_{b+\varepsilon}) + \frac{\varepsilon}{2}.$$

Since $t_n > 0$ it follows that $\tau_{b_n} \leq t_n$. In particular, we have

$$\limsup_{n\to\infty} \tau_{b_n} \leq \limsup_{n\to\infty} t_n = \tau_{b+\varepsilon}.$$

This shows the desired statement. □

REMARK 3.9. Let b be a boundary function. Since $\tau_b' \geq \tau_b$, if for $(X_t)_{t\geq 0}$ it holds that $\tau_b \overset{\mathrm{d}}{=} \tau_b'$, then we have in fact $\tau_b = \tau_b'$ almost surely.

LEMMA 3.10. *Let b be a boundary function. Then*

$$\lim_{\varepsilon\searrow 0} \tau_{b+\varepsilon} = \tau_b'$$

almost surely.

PROOF. Note that $\tau_{b+\varepsilon}$ is decreasing in ε and bounded from below by τ_b'. Thus the following limit exists and fulfills

$$\lim_{\varepsilon\searrow 0} \tau_{b+\varepsilon} \geq \tau_b'$$

almost surely. Thus it is left to show that $\lim_{\varepsilon\searrow 0} \tau_{b+\varepsilon} \leq \tau_b'$ almost surely. Without loss of generality we assume that $\tau_b' < \infty$. On this event we have that there exists a sequence $t_n \searrow \tau_b'$ (possibly random) with

$$X_{t_n} > b(t_n) \quad \forall n \in \mathbb{N}.$$

If $\tau_b' = 0$ this sequence fulfills $t_n > 0$ for every $n \in \mathbb{N}$. Let $\delta > 0$. Then there exists $n \in \mathbb{N}$ (possibly random) such that $t_n \leq \tau_b' + \delta$. Set

$$\varepsilon_n := \frac{1}{2}\left(X_{t_n} - b(t_n)\right) > 0$$

for which holds

$$X_{t_n} > b(t_n) + \varepsilon_n.$$

Therefore $\tau_{b+\varepsilon_n} \leq t_n \leq \tau_b' + \delta$. Thus we have that

$$\lim_{\varepsilon\searrow 0} \tau_{b+\varepsilon} \leq \tau_{b+\varepsilon_n} \leq \tau_b' + \delta.$$

Letting $\delta \searrow 0$ yields that $\lim_{\varepsilon\searrow 0} \tau_{b+\varepsilon} \leq \tau_b'$. This proves the desired statement. □

LEMMA 3.11. *Assume that $(X_t)_{t\geq 0}$ has right-continuous paths and is quasi-left-continuous. Let b be a boundary function and assume $\tau_b \overset{d}{=} \tau_b'$. Further, let $b_n \overset{\Gamma}{\to} b$. Then on $\{\liminf_{n\to\infty} \tau_{b_n} > 0\} \cup \{\tau_b = \overline{\tau_b}\}$ we have*

$$\lim_{n\to\infty} \tau_{b_n} = \tau_b$$

almost surely.

PROOF. By combining Lemma 3.6, Lemma 3.8, Lemma 3.10 and Remark 3.9 we have

$$\tau_b \leq \liminf_{n\to\infty} \tau_{b_n} \leq \limsup_{n\to\infty} \tau_{b_n} \leq \lim_{\varepsilon\searrow 0} \tau_{b+\varepsilon} = \tau_b$$

almost surely on $\{\liminf_{n\to\infty} \tau_{b_n} > 0\} \cup \{\tau_b = \overline{\tau_b}\}$, which concludes the proof. □

REMARK 3.12. If $b_n \xrightarrow{\Gamma} b$, in general it is not true that for every $s > 0$ also $b_n|_s \xrightarrow{\Gamma} b|_s$. The following Lemma 3.13 gives a sufficient condition on b such that the convergence is preserved. By Lemma A.4 we will see that we can find arbitrarily small $s > 0$ fulfilling this sufficient condition.

LEMMA 3.13. *Let $b_n \xrightarrow{\Gamma} b$ and $s > 0$. If $\liminf_{t \searrow s} b(t) = b(s)$, then $b_n|_s \xrightarrow{\Gamma} b|_s$.*

PROOF. If $t \in [0, s)$ for every sequence $t_n \to t$ we have

$$\lim_{n\to\infty} b_n|_s(t_n) = b|_s(t) = \infty.$$

If $t \in [s, \infty]$ and $t_n \to t$, then

$$\liminf_{n\to\infty} b_n|_s(t_n) \geq \liminf_{n\to\infty} b_n(t_n) \geq b(t) = b|_s(t).$$

Furthermore, due to the Γ-convergence for $t \in [s, \infty]$ there is a sequence $t_n \to t$ such that $b_n(t_n) \to b(t)$. If $t = s$, by Lemma 3.7 and our assumption, we can assume that $t_n \geq t = s$ for every n. Therefore we have

$$\lim_{n\to\infty} b_n|_s(t_n) = \lim_{n\to\infty} b_n(t_n) = b(t) = b|_s(t).$$

Hence it holds $b_n|_s \xrightarrow{\Gamma} b|_s$. □

LEMMA 3.14. *Let b be a boundary function. Then*

$$\tau_b = \lim_{s \searrow 0} \tau_{b|_s}$$

almost surely.

PROOF. The random variable $\tau_{b|_s}$ is monotone decreasing in s and bounded from below by τ_b. Thus the limit exists and we have $\tau_b \leq \lim_{s\searrow 0} \tau_{b|_s}$. Without loss of generality assume $\tau_b < \infty$. For $m \in \mathbb{N}$ there exists a time $t > 0$ (possibly random) such that $t \in [\tau_b, \tau_b + \frac{1}{m})$ and $X_t \geq b(t)$. Thus, if $r \in (0, t)$, then

$$X_t \geq b(t) = b|_r(t),$$

which means that $\tau_{b|_r} \leq t < \tau_b + \frac{1}{m}$. Consequently,

$$\tau_b \leq \lim_{s\searrow 0} \tau_{b|_s} \leq \tau_{b|_r} \leq \tau_b + \frac{1}{m}.$$

By $m \to \infty$ we obtain $\lim_{s\searrow 0} \tau_{b|_s} = \tau_b$ almost surely. □

REMARK 3.15. It is analogous to show that $\tau_b' = \lim_{s\searrow 0} \tau_{b|_s}'$ almost surely.

PROOF OF PROPOSITION 3.2. For $s > 0$ and $\varepsilon > 0$ we have that

$$\begin{aligned}
&\mathbb{P}\left(|\tau_{b_n} - \tau_b| > \varepsilon\right) \\
&\leq \mathbb{P}\left(|\tau_{b_n|_s} - \tau_{b_n}| > \frac{\varepsilon}{3}\right) + \mathbb{P}\left(|\tau_{b_n|_s} - \tau_{b|_s}| > \frac{\varepsilon}{3}\right) + \mathbb{P}\left(|\tau_{b|_s} - \tau_b| > \frac{\varepsilon}{3}\right) \\
&\leq \mathbb{P}\left(\tau_{b_n} \leq s\right) + \mathbb{P}\left(|\tau_{b_n|_s} - \tau_{b|_s}| > \frac{\varepsilon}{3}\right) + \mathbb{P}\left(|\tau_{b|_s} - \tau_b| > \frac{\varepsilon}{3}\right),
\end{aligned}$$

where we have used that

$$\mathbb{P}\left(|\tau_{b_n|_s}-\tau_{b_n}|>\frac{\varepsilon}{3}\right)\leq\mathbb{P}\left(\tau_{b_n|_s}\neq\tau_{b_n}\right)\leq\mathbb{P}\left(\tau_{b_n}\leq s\right).$$

Due to Lemma A.4 we can choose arbitrarily small $s>0$ such that $\liminf_{t\searrow s} b(t)=b(s)$. Lemma 3.13 shows that $b_n|_s\xrightarrow{\Gamma} b|_s$. Moreover, we have almost surely

$$\liminf_{n\to\infty}\tau_{b_n|_s}\geq s>0$$

and by assumption $\tau_{b|_s}\overset{\mathrm{d}}{=}\tau'_{b|_s}$. Hence we can apply Lemma 3.11 and obtain

$$\lim_{n\to\infty}\tau_{b_n|_s}=\tau_{b|_s}$$

almost surely. By Lemma 3.14 we have

$$\lim_{s\searrow 0}\tau_{b|_s}=\tau_b.$$

almost surely. Thus we have that

$$\begin{aligned}
&\limsup_{n\to\infty}\mathbb{P}\left(|\tau_{b_n}-\tau_b|>\varepsilon\right)\\
&\leq\lim_{s\searrow 0}\left(\limsup_{n\to\infty}\left(\mathbb{P}\left(\tau_{b_n}\leq s\right)+\mathbb{P}\left(|\tau_{b_n|_s}-\tau_{b|_s}|>\frac{\varepsilon}{3}\right)\right)+\mathbb{P}\left(|\tau_{b|_s}-\tau_b|>\frac{\varepsilon}{3}\right)\right)\\
&=\lim_{s\searrow 0}\limsup_{n\to\infty}\mathbb{P}\left(\tau_{b_n}\leq s\right)=0,
\end{aligned}$$

where the last equality comes from our assumption. This yields the statement. □

REMARK 3.16. Analogously as in Theorem 3.1 of [35] it can be shown that the Γ-convergence coincides with the convergence of the epigraphs in the Hausdorff metric. From Proposition 2.1.3 of [33] follows that every sequence of boundary functions has a convergent subsequence.

PROOF OF THEOREM 2.3. For $n\in\mathbb{N}$ let $t_k^n:=k2^{-n}$ with $k\in\mathbb{N}_0$. We will inductively define a boundary function b_n, which has only finite values at the discrete timepoints t_k^n. For $k\in\mathbb{N}$ let us assume $b_n(t_1^n),\ldots,b_n(t_{k-1}^n)$ are already defined. Since $\mathbb{P}\left(X_t\in\cdot\,\right)$ is diffuse, we can choose a value $b_n(t_k^n)\in[-\infty,\infty]$ such that

$$\mathbb{P}\left(X_{t_k^n}<b_n(t_k^n),\ldots,X_{t_1^n}<b_n(t_1^n)\right)=\mathbb{P}\left(\xi>t_k^n\right)$$

with $b_n(t_k^n)=-\infty$ if $\mathbb{P}\left(\xi>t_k^n\right)=0$. By setting $b_n(t)=\infty$ for all $t\notin\{k2^{-n}:k\in\mathbb{N}\}$, we obtain a lower semicontinuous function b_n. Note, that then by definition

$$\mathbb{P}\left(\tau_{b_n}>t\right)=\mathbb{P}\left(\xi>\lfloor t2^n\rfloor 2^{-n}\right),\qquad\forall t\geq 0.$$

This implies that

$$\tau_{b_n}\xrightarrow{\mathrm{d}}\xi$$

as $n\to\infty$. By the compactness of the set of boundary functions, see Remark 3.16, there is a lower semicontinuous function b, and a subsequence $N\subset\mathbb{N}$ such that

$$b_n\xrightarrow{\Gamma} b$$

along $n\in N$. By assumption we have $\tau_{b|_s}\overset{\mathrm{d}}{=}\tau'_{b|_s}$ for every $s>0$. Moreover, by assumption we have that $(X_t)_{t\geq 0}$ has right-continuous paths and is quasi-left-continuous. From Proposition 3.2 and Remark 3.3 we obtain that $\tau_{b_n}\xrightarrow{\mathbb{P}}\tau_b$ in probability along $n\in N$. This implies that $\tau_b\overset{\mathrm{d}}{=}\xi$. □

4. Uniqueness: Proof of Theorem 2.8. Let us explain beforehand the role of the conditions in the proof of Theorem 2.8. Conditions (U1) and (U2) allow to construct a boundary function which is a lower bound for any other solution. Conditions (E1), (E2), (E3) will yield that this lower bound is a solution. Condition (U3) will allow to infer that this lower bound is the unique solution.

Let $E \subseteq \mathbb{R}$ be an interval with $\overline{E} = [L, R]$ and $L, R \in [-\infty, \infty]$. Assume that $\mathbb{P}(X_t \in \cdot\,)$ is diffuse for any $t > 0$ and assume that $\mathbb{P}(\tau_R < \infty) = 0$.

Let $(t_k^n)_{n \in \mathbb{N}, k \in \{0,1,\dots,m_n\}} \subset [0, \infty)$ with $m_n \in \mathbb{N} \cup \{\infty\}$ be such that

$$0 = t_0^n < t_1^n < \dots t_{m_n}^n.$$

For $k \in \mathbb{N}$, such that $\mathbb{P}(\xi > t_k^n) > 0$ suppose $q_1^n, \dots q_{k-1}^n$ are already defined. Since $\mathbb{P}\left(X_{t_k^n} \in \cdot\,\right)$ is diffuse and $\mathbb{P}\left(X_{t_k^n} < R\right) = 1$ we can choose $q_k^n \in \overline{E}$ such that

$$\mathbb{P}\left(X_{t_k^n} < q_k^n, X_{t_{k-1}^n} < q_{k-1}^n, \dots, X_{t_1^n} < q_1^n\right) = \mathbb{P}(\xi > t_k^n)$$

and

$$\mathbb{P}\left(X_{t_k^n} < q, X_{t_{k-1}^n} < q_{k-1}^n, \dots, X_{t_1^n} < q_1^n\right) < \mathbb{P}(\xi > t_k^n)$$

for any $q < q_k^n$. Note that we have $q_k^n > \inf E = L$ since $\mathbb{P}(\xi > t_k^n) > 0$. For $k \in \mathbb{N}$ with $\mathbb{P}(\xi > t_k^n) = 0$ we set $q_k^n := \inf E = L$. By setting

$$b_n(t) := \begin{cases} q_k^n & : t = t_k^n, \\ \sup E = R & : t \notin \{t_k^n : k \in \mathbb{N}\}, \end{cases} \tag{4}$$

we obtain a boundary function b_n with values in $\overline{E}$. Note that by the definition of q_k^n we obtain that

$$b_n(t_k^n) = \sup \operatorname{supp}(\mathbb{P}\left(X_{t_k^n} \in \cdot \,\middle|\, \tau_{b_n} > t_k^n\right)).$$

REMARK 4.1. For Brownian motion on $\mathbb{R}$ this discretization appeared in [3] and [35], and implicitely in [22], [18]. In [35, Lemma 4.1] and [18, Theorem 5] it led to statements which are special cases of the following Lemma 4.2.

LEMMA 4.2. *Let $E \subseteq \mathbb{R}$ be an interval with $\overline{E} = [L, R]$. Let μ be a probability measure on E. Assume (U1), (U2) and (E1) and $\mathbb{P}(\tau_R < \infty) = 0$ with $\mathbb{P} := \mathbb{P}_\mu$. Let b be a boundary function with values in $\overline{E}$ such that $\tau_b \overset{d}{=} \xi$. Then for fixed $n \in \mathbb{N}$ we have*

$$\mathbb{P}\left(X_{t_k^n} \in \cdot \,\middle|\, \tau_{b_n} > t_k^n\right) \preceq_{\text{st}} \mathbb{P}\left(X_{t_k^n} \in \cdot \,\middle|\, \tau_b > t_k^n\right) \qquad \forall k \in \mathbb{N} : \mathbb{P}(\xi > t_k^n) > 0.$$

In particular, it follows that $b_n(t_k^n) \leq b(t_k^n)$ for all $k \in \mathbb{N}$.

In order to prove Lemma 4.2 we need one more tool. For a probability measure μ and $\alpha \in (0, 1]$ define for $A \subseteq \mathbb{R}$ measurable

$$T_\alpha(\mu)(A) := \frac{\mu(A \cap (-\infty, q_\alpha(\mu)])}{\mu((-\infty, q_\alpha(\mu)])},$$

where

$$q_\alpha(\mu) := \inf\{c \in \mathbb{R} : \mu((-\infty, c]) \geq \alpha\}.$$

The following statement is the one-sided version of Lemma 3.3 in [35]. In the presented generality we will use it in Section 5.

LEMMA 4.3. *Let μ, ν be probability measures and such that μ is diffuse. Then for $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 \le \alpha_2$ we have that $\mu \preceq_{\mathrm{st}} \nu$ implies $T_{\alpha_1}(\mu) \preceq_{\mathrm{st}} T_{\alpha_2}(\nu)$.*

PROOF. Since μ is diffuse we have that $\mu((-\infty, q_{\alpha_1}(\mu)]) = \alpha_1$. Assume that $\mu \preceq_{\mathrm{st}} \nu$. Then by the definitions we have $q_{\alpha_1}(\mu) \le q_{\alpha_2}(\nu)$ and it suffices to consider the case $c \le q_{\alpha_1}(\mu)$. Since $\nu((-\infty, q_{\alpha_2}(\nu)]) \ge \alpha_2$, we have

$$\begin{aligned} T_{\alpha_1}(\mu)((-\infty,c]) &= \frac{\mu((-\infty,c])}{\mu((-\infty,q_{\alpha_1}(\mu)])} = \frac{\mu((-\infty,c])}{\alpha_1} \ge \frac{\mu((-\infty,c])}{\alpha_2} \\ &\ge \frac{\mu((-\infty,c])}{\nu((-\infty,q_{\alpha_2}(\nu)])} \ge \frac{\nu((-\infty,c])}{\nu((-\infty,q_{\alpha_2}(\nu)])} = T_{\alpha_2}(\nu)((-\infty,c]). \end{aligned}$$

This shows $T_{\alpha_1}(\mu) \preceq_{\mathrm{st}} T_{\alpha_2}(\nu)$. □

For a probability measure μ we introduce the mapping P_t by

$$P_t(\mu) := \mathbb{P}_\mu\left(X_t \in \cdot\,\right). \tag{5}$$

PROOF OF LEMMA 4.2. Essentially, we follow the lines of the proof of Lemma 4.1 in [35], which was conducted in the case of reflected Brownian motion. Fix $n \in \mathbb{N}$. We abbreviate

$$\mu_k^n := \mathbb{P}\left(X_{t_k^n} \in \cdot \,\middle|\, \tau_{b_n} > t_k^n\right), \quad \mu_k := \mathbb{P}\left(X_{t_k^n} \in \cdot \,\middle|\, \tau_b > t_k^n\right).$$

Note that from the Markov property it follows that

$$\mu_k^n = T_{\alpha_k^n} \circ P_{t_k^n - t_{k-1}^n} \circ \ldots \circ T_{\alpha_1^n} \circ P_{t_1^n}(\mu), \tag{6}$$

where

$$\alpha_k^n = \frac{\mathbb{P}\left(\xi > t_k^n\right)}{\mathbb{P}\left(\xi > t_{k-1}^n\right)}.$$

We will prove the statement by induction over $k \in \mathbb{N}_0$ with $\mathbb{P}\left(\xi > t_k^n\right) > 0$, by comparing the mappings

$$H_k^n(\nu) := T_{\alpha_k^n} \circ P_{t_k^n - t_{k-1}^n}(\nu),$$

where ν is a probability measure on E, and

$$H_k(\nu) := \mathbb{P}_\nu\left(X_{t_k^n - t_{k-1}^n} \in \cdot \,\middle|\, \tau_{b^{t_{k-1}^n}} > t_k^n - t_{k-1}^n\right),$$

where we used the notation $b^s(t) := b(t+s)$ for $s > 0$. It follows by (6) and the Markov property that

$$H_k^n(\mu_{k-1}^n) = \mu_k^n \qquad \text{and} \qquad H_k(\mu_{k-1}) = \mu_k.$$

We now claim that we have

$$H_k^n(\mu_{k-1}) \preceq_{\mathrm{st}} H_k(\mu_{k-1}). \tag{7}$$

Using the Markov property we obtain

$$P_{t_k^n - t_{k-1}^n}(\mu_{k-1}) = \mathbb{P}_\mu\left(X_{t_k^n} \in \cdot \;\middle|\, \tau_b > t_{k-1}^n\right).$$

This shows that $P_{t_k^n - t_{k-1}^n}(\mu_{k-1})$ is diffuse and we have

$$P_{t_k^n - t_{k-1}^n}(\mu_{k-1})((-\infty, q_{\alpha_k^n}(P_{t_k^n - t_{k-1}^n}(\mu_{k-1}))]) = \alpha_k^n$$

and, by the Markov property and the fact that $\tau_b \overset{\mathrm{d}}{=} \xi$, we have

$$\mathbb{P}_{\mu_{k-1}}\left(\tau_{b^{t^n_{k-1}}} > t^n_k - t^n_{k-1}\right) = \alpha^n_k.$$

Therefore, if $c \leq q_{\alpha^n_k}(P_{t^n_k - t^n_{k-1}}(\mu_{k-1}))$, we have

$$\begin{aligned}
H^n_k(\mu_{k-1})((-\infty, c]) &= \frac{P_{t^n_k - t^n_{k-1}}(\mu_{k-1})((-\infty, c])}{\alpha^n_k} \\
&= \frac{\mathbb{P}_{\mu_{k-1}}\left(X_{t^n_k - t^n_{k-1}} \leq c\right)}{\alpha^n_k} \geq \frac{\mathbb{P}_{\mu_{k-1}}\left(X_{t^n_k - t^n_{k-1}} \leq c, \tau_{b^{t^n_{k-1}}} > t^n_k - t^n_{k-1}\right)}{\alpha^n_k} \\
&= \frac{\mathbb{P}_{\mu_{k-1}}\left(X_{t^n_k - t^n_{k-1}} \leq c, \tau_{b^{t^n_{k-1}}} > t^n_k - t^n_{k-1}\right)}{\mathbb{P}_{\mu_{k-1}}\left(\tau_{b^{t^n_{k-1}}} > t^n_k - t^n_{k-1}\right)} \\
&= \mathbb{P}_{\mu_{k-1}}\left(X_{t^n_k - t^n_{k-1}} \leq c \,\middle|\, \tau_{b^{t^n_{k-1}}} > t^n_k - t^n_{k-1}\right) = H_k(\mu_{k-1})((-\infty, c]).
\end{aligned}$$

This shows the claim. Now let us assume that $\mu^n_{k-1} \preceq_{\mathrm{st}} \mu_{k-1}$. Then Lemma 4.3 the fact that P_t preserves the usual stochastic order and the claim from (7) yield

$$\mu^n_k = H^n_k(\mu^n_{k-1}) \preceq_{\mathrm{st}} H^n_k(\mu_{k-1}) \preceq_{\mathrm{st}} H_k(\mu_{k-1}) = \mu_k.$$

Since $\mu^n_0 = \mu_0$ the desired ordering follows by induction. From the ordering it follows that $\mu^n_k((-\infty, c]) \geq \mu_k((-\infty, c])$ for all $c \in \mathbb{R}$, hence

$$b_n(t^n_k) = \sup \operatorname{supp}(\mu^n_k) \leq \sup \operatorname{supp}(\mu_k) \leq b(t^n_k)$$

for $k \in \mathbb{N}$ with $\mathbb{P}(\xi > t^n_k) > 0$. Since $b_n(t^n_k) = \inf E$ for $k \in \mathbb{N}$ with $\mathbb{P}(\xi > t^n_k) = 0$, the proof is finished. □

Let b be a boundary function with values in $\overline{E}$. We now introduce a discretization technique for b, which was already used in [22] and [13] for the case of Brownian motion. We use an adapted version. Let $D(b)$ be an arbitrary countable set and $D_n(b) \subset D_{n+1}(b)$ finite, such that $\bigcup_{n\in\mathbb{N}} D_n(b) = D(b)$. For $n \in \mathbb{N}$ define for every $k \in \mathbb{N}$

$$\tilde{t}^n_k := \inf\left\{ t \in [k2^{-n}, (k+1)2^{-n}] : b(t) = \inf_{s \in [k2^{-n}, (k+1)2^{-n}]} b(s) \right\}.$$

Set $A^1_n(b) := \{\tilde{t}^n_k : k \in \{1, 2, \ldots, n2^n\}\}\}$. Furthermore, let $(s_n)_{n\in\mathbb{N}}$ be an enumeration of $\{s \in [0, \infty) : \mathbb{P}(\tau_b = s) > 0\}$. Set $A^2_n(b) := \{s_1, \ldots, s_n\}$. Finally, set

$$A_n(b) := A^1_n(b) \cup A^2_n(b) \cup D_n(b).$$

By choosing $D_n(b) = \emptyset$ we end up with the construction used in [22]. Note that by $A^1_n(b) \subset A^1_{n+1}(b)$ we have

$$A_n(b) \subset A_{n+1}(b).$$

For $n \in \mathbb{N}$ let us define the boundary function

$$\hat{b}_n(t) := \begin{cases} b(t) & : t \in A_n(b), \\ R & : t \notin A_n(b). \end{cases} \tag{8}$$

LEMMA 4.4. *For a boundary function b it holds $\hat{b}_n \xrightarrow{\Gamma} b$ as $n \to \infty$.*

PROOF. Let $t \in [0,\infty]$. Assume $t_n \to t$. Then

$$\hat{b}_n(t_n) \geq b(t_n).$$

Thus,

$$\liminf_{n\to\infty} \hat{b}_n(t_n) \geq \liminf_{n\to\infty} b(t_n) \geq b(t).$$

For the second part of the Γ-convergence we distinguish two cases. Let us first assume that $t \in \bigcup_{n\in\mathbb{N}} A_n(b)$. Then for N large enough we have $t \in A_n(b)$ for all $n \geq N$. Hence,

$$\lim_{n\to\infty} \hat{b}_n(t) = \lim_{n\to\infty} b(t) = b(t).$$

Assume that $t \notin A_n(b)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be large enough and $k_n(t) \in \mathbb{N}$ such that

$$k_n(t)2^{-n} \leq t \leq (k_n(t)+1)2^{-n}.$$

Then we have

$$b(t) \geq \inf_{s\in[k_n(t)2^{-n},(k_n(t)+1)2^{-n}]} b(s) = b(\tilde{t}^n_{k_n(t)}).$$

Now define $t_n := \tilde{t}^n_{k_n(t)}$. We have $t_n \in A_n(b)$. It follows $t_n \to t$ and

$$\begin{aligned} b(t) &\geq \limsup_{n\to\infty} b(t_n) = \limsup_{n\to\infty} \hat{b}_n(t_n) \\ &\geq \liminf_{n\to\infty} \hat{b}_n(t_n) = \liminf_{n\to\infty} b(t_n) \geq b(t). \end{aligned}$$

This means that $\hat{b}_n(t_n) \to b(t)$. Altogether we obtain $\hat{b}_n \xrightarrow{\Gamma} b$. □

LEMMA 4.5. *Let b_1 and b_2 be boundary functions with values in $\overline{E}$ such that $\tau_{b_1} \overset{d}{=} \tau_{b_2} \overset{d}{=}: \xi$ and $b_1 \leq b_2$. Let $t \in (0, t^\xi)$ and assume*

$$b_2(t) = \sup\operatorname{supp}(\mathbb{P}(X_t \in \cdot\, | \tau_{b_2} > t)).$$

Then $b_1(t) = b_2(t)$.

PROOF. Assume $b_1(t) < b_2(t)$. Then by the assumption for the support we would have

$$\begin{aligned} \mathbb{P}(\xi > t) &= \mathbb{P}(\tau_{b_2} > t) = \mathbb{P}(\tau_{b_2} > t, X_t < b_2(t)) \\ &> \mathbb{P}(\tau_{b_2} > t, X_t < b_1(t)) \geq \mathbb{P}(\tau_{b_1} > t, X_t < b_1(t)) \\ &= \mathbb{P}(\tau_{b_1} > t) = \mathbb{P}(\xi > t). \end{aligned}$$

This contradiction shows $b_1(t) = b_2(t)$. □

LEMMA 4.6. *Let b^1 and b^2 be boundary functions and $b^1_n \xrightarrow{\Gamma} b^1$ and $b^2_n \xrightarrow{\Gamma} b^2$. Assume that $b^1_n \leq b^2_n$. Then $b^1 \leq b^2$.*

PROOF. Let $t \in [0,\infty]$. Let $t_n \to t$ such that $b^2_n(t_n) \to b^2(t)$. Then

$$b^2(t) = \lim_{n\to\infty} b^2(t_n) \geq \liminf_{n\to\infty} b^1(t_n) \geq b^1(t).$$

This finishes the proof. □

PROOF OF THEOREM 2.8. Let b, β be boundary functions with values in $\overline{E}$ such that $\tau_b \overset{\mathrm{d}}{=} \tau_\beta \overset{\mathrm{d}}{=} \xi$. Recall the construction of $\hat{b}_n$ and $\hat{\beta}_n$ from (8) for b and β, respectively. In particular, recall $A_n^1(b), A_n^2(b)$ and $A_n^1(\beta), A_n^2(\beta)$. We can choose

$$D_n(b) \coloneqq A_n^1(\beta) \cup A_n^2(\beta), \qquad D_n(\beta) \coloneqq A_n^1(b) \cup A_n^2(b).$$

This means that in the construction from (8) we have

$$A_n \coloneqq A_n(b) = A_n(\beta).$$

Note that thus $\hat{b}_n$ depends also on β and $\hat{\beta}_n$ on b. Now, since $\mathbb{P}(X_t \in \cdot\,)$ is diffuse we can construct the boundary function b_n from the construction (4), where we choose

$$\{t_0^n, t_1^n, \ldots, t_{m_n}^n\} = \{0\} \cup A_n.$$

Due to the Markov property and the stochastic order preservation we can apply Lemma 4.2 for the solutions b and β separately but with the same set A_n of discrete timesteps. With recalling (8) this leads to

$$b_n(t) \le b(t) = \hat{b}_n(t) \quad \text{and} \quad b_n(t) \le b(t) = \hat{\beta}_n(t) \qquad \forall t \in A_n.$$

This means that $b_n \le \hat{b}_n$ and $b_n \le \hat{\beta}$ altogether. Now note that, by Remark 3.16, there is a boundary function b^+ and a subsequence $N \subset \mathbb{N}$ such that along N

$$b_n \overset{\Gamma}{\to} b^+.$$

By Lemma 4.4 we have that

$$\hat{b}_n \overset{\Gamma}{\to} b \quad \text{and} \quad \hat{\beta}_n \overset{\Gamma}{\to} \beta.$$

Thus we have by Lemma 4.6 that

$$b^+ \le b \quad \text{and} \quad b^+ \le \beta.$$

By the definition of b_n from (4) under $\mathbb{P}$ we have on the one hand that

$$\tau_{b_n} \overset{\mathrm{d}}{\to} \xi$$

as $n \to \infty$. By assumption we have $\tau_{b^+|s} = \tau'_{b^+|s}$ $\mathbb{P}$-a.s. for $s > 0$ and $(X_t)_{t \ge 0}$ has $\mathbb{P}$-a.s. right-continuous paths and is quasi-left-continuous. By Proposition 3.2 and Remark 3.3 we obtain

$$\tau_{b_n} \overset{\mathbb{P}}{\to} \tau_{b^+}$$

in probability. This means that $\tau_{b^+} \overset{\mathrm{d}}{=} \xi$ under $\mathbb{P}$. Since b_n has values in $\overline{E}$, by the definition of the Γ-convergence, it follows that b^+ is a boundary function with values in $\overline{E}$. By assumption we have

$$b(t) = \sup \operatorname{supp}\left(\mathbb{P}\left(X_t \in \cdot\,|\tau_b > t\right)\right), \beta(t) = \sup \operatorname{supp}\left(\mathbb{P}\left(X_t \in \cdot\,|\tau_\beta > t\right)\right)$$

for $t \in I^\xi$. Hence Lemma 4.5 yields that

$$b(t) = b^+(t) = \beta(t)$$

for every $t \in I^\xi$. □

5. Comparison principle: Proof of Theorem 2.11. The following proof essentially follows the lines of the proof of Theorem 2.2 of [35]. Due to its brevity we include it for completeness.

PROOF OF THEOREM 2.11. Let $t_k^n := k2^{-n}$ with $k \in \mathbb{N}_0$. For the measure $\mathbb{P}_{\mu_i}$ and the random variable ξ_i let b_n^i be the sequence of boundary function constructed in (4). For $k \in \mathbb{N}$ with $\mathbb{P}(\xi_i > t_k^n) > 0$ let

$$\alpha_k^{i,n} := \frac{\mathbb{P}(\xi_i > t_k^n)}{\mathbb{P}\left(\xi_i > t_{k-1}^n\right)}.$$

Since $\xi_1 \preceq_{\mathrm{hr}} \xi_2$ we have that $\alpha_k^{1,n} \leq \alpha_k^{2,n}$. Recall the mapping $P_t(\mu) = \mathbb{P}_\mu(X_t \in \cdot\,)$ from (5). Since P_t preserves the order $\preceq_{\mathrm{st}}$ and $\mathbb{P}_{\mu_i}(X_t \in \cdot\,)$ are diffuse measures, we obtain by Lemma 4.3 that

$$\begin{aligned}\mathbb{P}_{\mu_1}\left(X_{t_k^n} \in \cdot \,\middle|\tau_{b_n^1} > t_k^n\right) &= T_{\alpha_k^{1,n}} \circ P_{t_k^n - t_{k-1}^n} \circ \ldots T_{\alpha_1^{1,n}} \circ P_{t_1^n}(\mu_1)\\ &\preceq_{\mathrm{st}} T_{\alpha_k^{2,n}} \circ P_{t_k^n - t_{k-1}^n} \circ \ldots T_{\alpha_1^{2,n}} \circ P_{t_1^n}(\mu_2) = \mathbb{P}_{\mu_2}\left(X_{t_k^n} \in \cdot \,\middle|\tau_{b_n^2} > t_k^n\right).\end{aligned}$$

This implies

$$\begin{aligned}b_n^1(t_k^n) &= \sup \operatorname{supp}(\mathbb{P}_{\mu_1}\left(X_{t_k^n} \in \cdot \,\middle|\tau_{b_n^1} > t_k^n\right))\\ &\leq \sup \operatorname{supp}(\mathbb{P}_{\mu_2}\left(X_{t_k^n} \in \cdot \,\middle|\tau_{b_n^2} > t_k^n\right)) = b_n^2(t_k^n)\end{aligned}$$

for $k \in \mathbb{N}$ with $\mathbb{P}(\xi_1 > t_k^n) > 0$. Since $t^{\xi_1} \leq t^{\xi_2}$ this means that $b_n^1 \leq b_n^2$. Now let b_1, b_2 be accumulation points of the sequences $(b_n^1)_{n\in\mathbb{N}}$ and $(b_n^2)_{n\in\mathbb{N}}$ such that $N \subseteq \mathbb{N}$ is a subsequence with $b_n^i \xrightarrow{\Gamma} b_i$ along N. Lemma 4.6 implies that

$$b_1 \leq b_2.$$

As in the proof of Theorem 2.8 we have that Proposition 3.2 implies that $\tau_{b_i} \stackrel{\mathrm{d}}{=} \xi_i$ under $\mathbb{P}_{\mu_i}$. □

6. Conditions for Lévy processes: Proof of Theorem 2.13. In this section let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$, where we allow $\mathbb{P}(X_0 \in \cdot\,)$ to be an arbitrary probability measure on $\mathbb{R}$. We will show that under suitable conditions $(X_t)_{t\geq 0}$ fulfills the conditions of Theorem 2.8, under which we established existence and uniqueness for the inverse first-passage time problem. This leads to the proof of Theorem 2.13, which is to be found at the end of the section. At first, we will collect the essential steps in preliminary statements. We begin with the fact that (E1) already implies (E3) for Lévy processes.

PROPOSITION 6.1. *Let $(X_t)_{t\geq 0}$ be a Lévy process such that $\mathbb{P}(X_1 \in \cdot\,)$ is diffuse. Let b be a boundary function. Then*

$$\tau_b = \tau_b'$$

almost surely.

The key idea for the proof of the statement is taken from Lemma 6.2 of [22], where the statement was proved for Brownian motion in an a very similiar manner. For diffusions on $\mathbb{R}$ a corresponding statement was shown in [13].

PROOF. As first step we will assume that $b = b|_s$ for some $s > 0$. Since X_s is independent from the future increments and its law is diffuse, we have for $t \geq s$ that the law of

$$Z_t := \sup_{r\in[s,t]} (X_r - b(r)) = X_s + \sup_{r\in[s,t]} (X_r - X_s - b(r))$$

is diffuse. In particular it holds $\mathbb{P}(Z_t = 0) = 0$. Recall that for a process with right-continuous paths we have $X_{\tau_b} \geq b(\tau_b)$ almost surely. Moreover, since $\inf_{t\in[0,s)} b(t) = \infty$ it holds that $\tau_b \geq s$. For $t \geq s$ we have

$$\{\tau_b \leq t\} \subseteq \{Z_t \geq 0\}, \quad \{Z_t > 0\} \subseteq \{\tau_b' \leq t\}.$$

Consequently, we have for $t \geq s$ that

$$\begin{aligned}\mathbb{P}(\tau_b \leq t) &= \mathbb{P}(\tau_b \leq t, Z_t \geq 0) = \mathbb{P}(\tau_b \leq t, Z_t > 0)\\ &= \mathbb{P}\left(\tau_b \leq t, Z_t > 0, \tau_b' \leq t\right) = \mathbb{P}\left(\tau_b \leq t, Z_t \geq 0, \tau_b' \leq t\right)\\ &= \mathbb{P}\left(\tau_b \leq t, \tau_b' \leq t\right) = \mathbb{P}\left(\tau_b' \leq t\right).\end{aligned}$$

This shows $\tau_b \overset{\mathrm{d}}{=} \tau_b'$.

For an arbitrary boundary function b and $s > 0$, since $\tau_{b|_s} \overset{\mathrm{d}}{=} \tau_{b|_s}'$ and $\tau_{b|_s} \leq \tau_{b|_s}'$, it follows that $\tau_{b|_s} = \tau_{b|_s}'$ almost surely. Lemma 3.14 and Remark 3.15 yield

$$\tau_b = \lim_{s\searrow 0} \tau_{b|_s} = \lim_{s\searrow 0} \tau_{b|_s}' = \tau_b'$$

almost surely. This finishes the proof. □

For a measure μ on $\mathbb{R}$ we denote with $\overline{\operatorname{supp}}(\mu)$ the closure of $\operatorname{supp}(\mu)$ in $[-\infty,\infty]$. Recall the definition of a characteristic triple of a Lévy process in (1).

PROPOSITION 6.2. *Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic triple (a, σ^2, Π) and $X_0 = 0$.*

(I) *If $0 \in \operatorname{supp}(\Pi)$ and $(X_t)_{t\geq 0}$ is a subordinator without drift, then for every boundary function $b : [0,\infty] \to [0,\infty]$ and $t > 0$ with $\mathbb{P}(\tau_b > t) > 0$ we have*

$$\overline{\operatorname{supp}}(\mathbb{P}(X_t \in \cdot \mid \tau_b > t)) = [0, b(t)].$$

(II) *If $0 \in \operatorname{supp}(\Pi)$ and $(-X_t)_{t\geq 0}$ is a subordinator without drift, then for every boundary function $b : [0,\infty] \to [-\infty, 0]$ with $\mathbb{P}(\tau_b > 0) > 0$ and $t > 0$ with $b(t) \leq b(u)$ for all $u \in [0,t]$ we have*

$$\operatorname{supp}(\mathbb{P}(X_t \in \cdot \mid \tau_b > t)) = (-\infty, b(t)].$$

(III) *If one of the following holds,*
   (i) *$(X_t)_{t\geq 0}$ has unbounded variation, i.e. $\sigma \neq 0$ or $\int_{\mathbb{R}}(1 \wedge |x|)\Pi(\mathrm{d}x) = \infty$,*
   (ii) *$0 \in \operatorname{supp}(\Pi)$ and $\Pi((-\infty,0)) > 0$ and $\Pi((0,\infty)) > 0$,*

   then for every boundary function $b : [0,\infty] \to [-\infty,\infty]$ and $t > 0$ with $\mathbb{P}(\tau_b > t) > 0$ we have

$$\overline{\operatorname{supp}}(\mathbb{P}(X_t \in \cdot \mid \tau_b > t)) = (-\infty, b(t)].$$

The idea of the proof for Proposition 6.2 is to use the Lévy-Itô decomposition and extract suitable components of the process which lead the path into desired regions with positive probability. This is inspired by Chapter 5 of [43, p.148]. In order to do so we will have to make a case distinction since the suitable components of the process differ from case to case.

We will work with the following general decomposition, which then is specified in the case distinctions. Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic triple (a,σ^2,Π) and $X_0=0$. Let Π_1 and Π_2 be measures on $\mathbb{R}$ such that $\Pi=\Pi_1+\Pi_2$. Let $\eta\in(0,1)$. We decompose formally

$$X_t = Y_t - a't + \sigma B_t + P_t^\eta + M_t^\eta, \tag{9}$$

where

$$a' = a + \int_{(-1,1)} x\Pi_1(\mathrm{d}x) + \int_{(-1,1)\setminus(-\eta,\eta)} x\Pi_2(dx)$$

and $(B_t)_{t\geq 0}$ is a standard Brownian motion and $(Y_t)_{t\geq 0}$, $(P_t^\eta)_{t\geq 0}$ and $(M_t^\eta)_{t\geq 0}$ are Lévy processes such that

$$-\log\left(\mathbb{E}\left[e^{i\theta Y_1}\right]\right) = \int_{\mathbb{R}} (1-e^{i\theta x})\Pi_1(\mathrm{d}x)$$

and

$$-\log\left(\mathbb{E}\left[e^{i\theta P_1^\eta}\right]\right) = \int_{\mathbb{R}\setminus(-\eta,\eta)} (1-e^{i\theta x})\Pi_2(\mathrm{d}x)$$

and

$$-\log\left(\mathbb{E}\left[e^{i\theta M_1^\eta}\right]\right) = \int_{(-\eta,\eta)} (1-e^{i\theta x}+i\theta x)\Pi_2(\mathrm{d}x).$$

Note that P_t^η is a compound Poisson process and M_t^η is a zero-mean square-integrable martingale with $\mathbb{E}\left[M_t^\eta)^2\right] = t\int_{(-\eta,\eta)} x^2\Pi(dx)$. This means that by Doob's inequality for every $t, C>0$ we can choose $\eta>0$ such that

$$\mathbb{P}\left(\sup_{s\leq t}|M_s^\eta|\geq C\right) \leq \frac{t}{C}\int_{(-\eta,\eta)} x^2\Pi_2(dx) < 1. \tag{10}$$

For treating (III) in Proposition 6.2 we will use the following auxiliary lemma.

LEMMA 6.3. *Let $(X_t)_{t\geq 0}$ be a Markov process and μ a probability measure on $\mathbb{R}$. Assume that for any $t>0$ and for any $K>x>0$ we have that*

$$\operatorname{supp}(\mathbb{P}_x(X_t\in\cdot,\tau_K>t)) = (-\infty,K].$$

Then we have for any boundary function $b:[0,\infty]\to[-\infty,\infty]$ and $t>0$ with $\mathbb{P}_\mu(\tau_b>t)>0$ that

$$\overline{\operatorname{supp}}(\mathbb{P}_\mu(X_t\in\cdot\,|\tau_b>t)) = (-\infty,b(t)].$$

PROOF. Let b and $t>0$ as in the statement. We abbreviate $\mathbb{P}=\mathbb{P}_\mu$. It holds $b(t)>-\infty$ since $\mathbb{P}(\tau_b>t)>0$. Define for $0<\delta<t$

$$K_\delta := \inf_{s\in[t-\delta,t]} b(s).$$

Furthermore, since b is lower semicontinuous and $\mathbb{P}(\tau_b>t)>0$, we have for $0<r<t-\delta$ that

$$K_1 := \inf_{s\in[r,t-\delta]} b(s) = \min_{s\in[r,t-\delta]} b(s) > -\infty.$$

Let $s\in[r,t-\delta]$ such that $b(s)=K_1$. Define $\mu_s := \mathbb{P}(X_s\in\cdot,\tau_b>s)$. Note that

$$\emptyset \neq \operatorname{supp}(\mu_s) \subseteq (-\infty,b(s)] = (-\infty,K_1]$$

but $\mu_s(\{K_1\}) = 0$. We can write

$$\begin{aligned}\mu_{t-\delta} &:= \mathbb{P}_{\mu_s}\left(X_{t-\delta-s} \in \cdot\,, \tau_{K_1} > t-\delta-s\right)\\ &= \int_{(-\infty,K_1)} \mathbb{P}_x\left(X_{t-\delta-s} \in \cdot\,, \tau_{K_1} > t-\delta-s\right)\mu_s(\mathrm{d}x).\end{aligned}$$

The assumption of the statement ensures that

$$(-\infty, K_1) \subseteq \operatorname{supp}(\mathbb{P}_x\left(X_{t-\delta-s} \in \cdot\,, \tau_{K_1} > t-\delta-s\right))$$

for every $x < K_1$, which implies

$$\overline{\operatorname{supp}}(\mu_{t-\delta}) = (-\infty, K_1]$$

with $\mu_{t-\delta}(\{K_1\}) = 0$. Now let $z \in (-\infty, K_\delta)$ and $\varepsilon \in (0, K_\delta - z)$. Note that due to the assumption of the statement we have that

$$(-\infty, K_\delta) \subseteq \operatorname{supp}(\mathbb{P}_x\left(X_\delta \in \cdot, \tau_{K_\delta} > \delta\right))$$

for every $x < K_\delta$. Thus, using the Markov property, we have

$$\begin{aligned}&\mathbb{P}\left(X_t \in (z-\varepsilon, z+\varepsilon), \tau_b > t\right)\\ &\geq \mathbb{P}\left(X_t \in (z-\varepsilon, z+\varepsilon), \tau_b > s, \tau_{K_1} \notin [s, t-\delta], \tau_{K_\delta} \notin [t-\delta, t]\right)\\ &= \int_{(-\infty,K_1)} \mathbb{P}_x\left(X_\delta \in (z-\varepsilon, z+\varepsilon), \tau_{K_\delta} > \delta\right)\mu_{t-\delta}(\mathrm{d}x) > 0.\end{aligned}$$

This means that

$$(-\infty, K_\delta) \subseteq \operatorname{supp}(\mathbb{P}\left(X_t \in \cdot\,, \tau_b > t\right)).$$

But since $K_\delta \to b(t)$ as $\delta \to 0$, we have that

$$\overline{\operatorname{supp}}(\mathbb{P}\left(X_t \in \cdot\,, \tau_b > t\right)) = (-\infty, b(t)].$$

This completes the proof. □

Let us establish conditions which imply the condition of the auxiliary lemma.

LEMMA 6.4. *Let $(X_t)_{t\geq 0}$ be a Lévy process with a characteristic triple (a, σ^2, Π) and $X_0 = 0$. If one of the following holds,*

(i) *$(X_t)_{t\geq 0}$ has unbounded variation, i.e. $\sigma \neq 0$ or $\int_{\mathbb{R}}(1 \wedge |x|)\Pi(\mathrm{d}x) = \infty$,*
(ii) *$0 \in \operatorname{supp}(\Pi)$ and $\Pi((-\infty, 0)) > 0$ and $\Pi((0,\infty)) > 0$,*

then for $K > 0$ we have

$$\operatorname{supp}(\mathbb{P}\left(X_t \in \cdot\ |\tau_K > t\right)) = (-\infty, K].$$

We will prove this lemma by using components which have the following form.

LEMMA 6.5. *Let $(X_t)_{t\geq 0}$ be a Lévy process with a characteristic triple (a, σ^2, Π) and $X_0 = 0$. If $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$ and for $\gamma := -\left(a + \int_{(-1,1)} x\Pi(\mathrm{d}x)\right)$ one of the following conditions is fulfilled,*

(a) *$\gamma \leq 0$ and $0 \in \operatorname{supp}(\Pi(\cdot \cap (0,\infty)))$,*
(b) *$\gamma \geq 0$ and $0 \in \operatorname{supp}(\Pi(\cdot \cap (-\infty, 0)))$,*
(c) *$0 \in \operatorname{supp}(\Pi)$ and $\Pi((-\infty,0)) > 0$ and $\Pi((0,\infty)) > 0$,*

then for $K > 0$ we have

$$\operatorname{supp}(\mathbb{P}(X_t \in \cdot \ |\tau_K > t)) = \begin{cases} (\gamma t, K] & : \textit{(a)}, \\ (-\infty, \min(K, \gamma t)] & : \textit{(b)}, \\ (-\infty, K] & : \textit{(c)}. \end{cases}$$

PROOF. For $\Pi_1 := \Pi$ and $\gamma = -a'$ the decomposition of (9) reduces to

$$X_t = Y_t + \gamma t$$

For $c_1 < c_2$ and $\kappa_1 < \kappa_2$ let us define

$$Y_t^c := \int_0^t (X_s - X_{s-}) \,\mathrm{d}N_s^c, \qquad Y_t^\kappa := \int_0^t (X_s - X_{s-}) \,\mathrm{d}N_s^\kappa,$$

into independent processes, where

$$N_t^c := \sum_{s \le t} \mathbb{1}_{(c_1, c_2)}(X_s - X_{s-}), \qquad N_t^\kappa := \sum_{s \le t} \mathbb{1}_{(\kappa_1, \kappa_2)}(X_s - X_{s-})$$

are Poisson processes with intensities $\Pi((c_1, c_2))$ and $\Pi((\kappa_1, \kappa_2))$, respectively.

Assume condition (a). Let $x \in (\gamma t, K)$ and let $\varepsilon > 0$ such that

$$\varepsilon < \min\{x - \gamma t, K - x\}.$$

Since $0 \in \operatorname{supp}(\Pi(\cdot \ \cap (0, \infty)))$ there is $\kappa \in (0, \varepsilon/2) \cap \operatorname{supp}(\Pi)$. Let $\delta \in (0, t)$ so that

$$-\gamma\delta < \frac{\varepsilon}{2}.$$

Since $\kappa < \varepsilon/2$ and $\gamma(t - \delta) < x - \varepsilon/2$ there is $n_\kappa \in \mathbb{N}$ such that

$$\gamma(t - \delta) + n_\kappa \cdot \kappa \in \left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right).$$

There are $0 < \kappa_1 < \kappa < \kappa_2 < \varepsilon/2$ such that

$$\gamma(t - \delta) + n_\kappa \cdot (\kappa_1, \kappa_2) \subseteq \left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right).$$

Since $\kappa \in \operatorname{supp}(\Pi)$ we have that $\Pi((\kappa_1, \kappa_2)) > 0$. With the decomposition

$$Y_t = Y_t^0 + Y_t^\kappa$$

we observe that

$$\begin{aligned} &\{Y_s = 0 \,\forall s \le t - \delta\} \cap \left\{Y_s^0 = 0 \,\forall s \le t\right\} \cap \{N_t^\kappa = n_\kappa\} \\ &\subseteq \{Y_t + \gamma t \in (x - \varepsilon, x + \varepsilon)\} \cap \left\{\sup_{s \le t}(Y_s + \gamma s) < K\right\}. \end{aligned}$$

By independence, the Markov property, the fact that the intensity of $(Y_s)_{s \ge 0}$ and $(Y_s^0)_{s \ge 0}$ is finite and that $\Pi((\kappa_1, \kappa_2)) > 0$, we have that

$$\begin{aligned} &\mathbb{P}\left(Y_s = 0 \,\forall s \le t - \delta, Y_s^0 = 0 \,\forall s \le t, N_t^\kappa = n_\kappa\right) \\ &= \mathbb{P}(Y_s = 0 \,\forall s \le t - \delta)\, \mathbb{P}\left(Y_s^0 = 0 \,\forall s \le \delta\right) \mathbb{P}(N_\delta^\kappa = n_\kappa) > 0. \end{aligned}$$

This means that in the situation of (a) we have that

$$\operatorname{supp}(\mathbb{P}(X_t \in \cdot\,, \tau_K > t)) = [\gamma t, K].$$

Assume condition (b). Let $x \in (-\infty, \min(K, \gamma t))$ and let $\varepsilon < \min(K, \gamma t) - x$. Let $\delta \in (0, t)$ such that

$$\gamma\delta < K.$$

Since $0 \in \operatorname{supp}(\Pi(\cdot \cap (-\infty, 0)))$ there is $\kappa \in (-\varepsilon, 0) \cap \operatorname{supp}(\Pi)$. Since $|\kappa| < \varepsilon$ and $x + \varepsilon < \gamma t$ there is $n_\kappa \in \mathbb{N}$ such that

$$\gamma t + n_\kappa \cdot \kappa \in (x - \varepsilon, x + \varepsilon).$$

There are $-\varepsilon < \kappa_1 < \kappa < \kappa_2 < 0$ such that

$$\gamma t + n_\kappa \cdot (\kappa_1, \kappa_2) \subseteq (x - \varepsilon, x + \varepsilon).$$

Since $\kappa \in \operatorname{supp}(\Pi)$ we have that $\Pi((\kappa_1, \kappa_2)) > 0$. With the decomposition

$$Y_t = Y_t^0 + Y_t^\kappa$$

we observe that

$$\begin{aligned}&\left\{Y_s^0 = 0 \,\forall s \le t\right\} \cap \{N_\delta^\kappa = n_\kappa\} \cap \{N_s^\kappa = n_\kappa \,\forall \delta \le s \le t\}\\ &\subseteq \left\{ Y_t + \gamma t \in (x - \varepsilon, x + \varepsilon), \sup_{s \le t}(Y_s + \gamma s) < K \right\}.\end{aligned}$$

By independence, the Markov property, the fact that the intensity of $(Y_s)_{s \ge 0}$ and $(Y_s^0)_{s \ge 0}$ is finite and that $\Pi((\kappa_1, \kappa_2)) > 0$, we have that

$$\begin{aligned}&\mathbb{P}\left(Y_s^0 = 0 \,\forall s \le t, N_\delta^\kappa = n_\kappa, N_s^\kappa = n_\kappa \,\forall \delta \le s \le t\right)\\ &= \mathbb{P}\left(Y_s^0 = 0 \,\forall s \le t\right) \mathbb{P}\left(N_\delta^\kappa = n_\kappa\right) \mathbb{P}\left(N_s^\kappa = 0 \,\forall 0 \le s \le t - \delta\right) > 0.\end{aligned}$$

This means that in the situation of (b) we have that

$$\operatorname{supp}(\mathbb{P}(X_t \in \cdot\,, \tau_K > t)) = (-\infty, \min(K, \gamma t)].$$

Assume the case (c). We have that

$$\text{(c.1) } 0 \in \operatorname{supp}(\Pi(\cdot \cap (0, \infty))) \quad \text{or} \quad \text{(c.2) } 0 \in \operatorname{supp}(\Pi(\cdot \cap (-\infty, 0))).$$

Suppose that (c.1) holds: Let $x \in (-\infty, K)$ and $\varepsilon > 0$ such that

$$\varepsilon < K - x.$$

Let $\delta_1, \delta_2 \in (0, t)$ such that

$$\gamma\delta_1 < K \quad \text{and} \quad |\gamma\delta_2| < \frac{\varepsilon}{2}.$$

By condition (c) there is $c \in (-\infty, 0) \cap \operatorname{supp}(\Pi)$. Thus there is $m_c \in \mathbb{N}$ such that

$$m_c \cdot c + \gamma(t - \delta_2) < x - \frac{\varepsilon}{2}.$$

Since $0 \in \operatorname{supp}(\Pi(\cdot \cap (0, \infty)))$ there is $\kappa \in (0, \varepsilon) \cap \operatorname{supp}(\Pi)$. Since $\kappa < \varepsilon$ and $m_c \cdot c + \gamma(t - \delta_2) < x - \varepsilon/2$ there is $n_\kappa \in \mathbb{N}$ such that

$$m_c \cdot c + \gamma(t - \delta_2) + n_\kappa \cdot \kappa \in \left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right).$$

There are $c_1 < c < c_2 < 0$ and $0 < \kappa_1 < \kappa < \kappa_2 < \varepsilon$ such that

$$m_c \cdot (c_1, c_2) + \gamma(t - \delta_2) + n_\kappa \cdot (\kappa_1, \kappa_2) \subseteq \left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right).$$

Since $c,\kappa \in \operatorname{supp}(\Pi)$ we have that $\Pi((c_1,c_2))>0$ and $\Pi((\kappa_1,\kappa_2))>0$. Now observe that

$$\begin{aligned}&\left\{Y_s^0=0\,\forall s\le t\right\}\cap\{N_s^c=m_c\,\forall \delta_1\le s\le t\}\cap\{N_s^\kappa=0\,\forall s\le t-\delta_2\}\cap\{N_t^\kappa=n_\kappa\}\\&\subseteq\left\{Y_t+\gamma t\in(x-\varepsilon,x+\varepsilon),\sup_{s\le t}(Y_s+\gamma s)<K\right\}.\end{aligned}$$

By independence, the Markov property and the fact that the intensities of $(Y_s^0)_{s\ge0}$, $(Y_s^\eta)_{s\ge0}$ and $(Y_s^c)_{s\ge0}$ are finite and for $(Y_s^\eta)_{s\ge0}$ and $(Y_s^c)_{s\ge0}$ even positive, we have that

$$\begin{aligned}&\mathbb{P}\left(Y_s^0=0\,\forall s\le t, N_s^c=m_c\,\forall\delta_1\le s\le t, N_s^\kappa=0\,\forall s\le t-\delta_2, N_t^\kappa=n_\kappa\right)\\&=\mathbb{P}\left(Y_s^0=0\,\forall s\le t\right)\mathbb{P}\left(N_{\delta_1}^c=m_c\right)\mathbb{P}\left(N_s^c=0\,\forall s\le t-\delta_1\right)\\&\qquad\cdot\mathbb{P}\left(N_s^\kappa=0\,\forall s\le t-\delta_2\right)\mathbb{P}\left(N_{\delta_2}^\kappa=n_\kappa\right)>0.\end{aligned}$$

This finishes the proof of the lemma for the case (c.1).

Suppose that (c.2) holds: Let $x\in(-\infty,K)$ and $\varepsilon>0$ such that

$$\varepsilon<K-x.$$

Let $\delta_1,\delta_2\in(0,t)$ such that

$$\gamma\delta_1<K\quad\text{and}\quad|\gamma\delta_2|<\frac{\varepsilon}{2}.$$

By condition (c) there is $c\in(0,\infty)\cap\operatorname{supp}(\Pi)$. Thus there is $m_c\in\mathbb{N}$ such that

$$m_c\cdot c+\gamma(t-\delta_2)>x+\frac{\varepsilon}{2}.$$

Since $0\in\operatorname{supp}(\Pi(\cdot\,\cap(-\infty,0)))$ there is $\kappa\in(-\varepsilon,0)\cap\operatorname{supp}(\Pi)$. Since $|\kappa|<\varepsilon$ and $m_c\cdot c+\gamma(t-\delta_2)>x+\varepsilon/2$ there is $n_\kappa\in\mathbb{N}$ such that

$$m_c\cdot c+\gamma(t-\delta_2)+n_\kappa\cdot\kappa\in\left(x-\frac{\varepsilon}{2},x+\frac{\varepsilon}{2}\right)$$

There are $0<c_1<c<c_2$ and $-\varepsilon<\kappa_1<\kappa<\kappa_2<0$ such that

$$m_c\cdot(c_1,c_2)+\gamma(t-\delta_2)+n_\kappa\cdot(\kappa_1,\kappa_2)\subseteq\left(x-\frac{\varepsilon}{2},x+\frac{\varepsilon}{2}\right)$$

Since $c,\kappa\in\operatorname{supp}(\Pi)$ we have $\Pi((c_1,c_2))>0$ and $\Pi((\kappa_1,\kappa_2))>0$. By decomposing

$$Y_t=Y_t^0+Y_t^c+Y_t^\kappa$$

we observe that

$$\begin{aligned}&\left\{Y_s^0=0\,\forall s\le t\right\}\cap\{N_s^\kappa=n_\kappa\,\forall \delta_1\le s\le t\}\cap\{N_s^c=0\,\forall s\le t-\delta_2\}\cap\{N_t^c=m_c\}\\&\subseteq\left\{Y_t+\gamma t\in(x-\varepsilon,x+\varepsilon),\sup_{s\le t}(Y_s+\gamma s)<K\right\}.\end{aligned}$$

By independence, the Markov property and the fact that the intensities of $(Y_s^0)_{s\ge0}$, $(Y_s^\kappa)_{s\ge0}$ and $(Y_s^c)_{s\ge0}$ are finite and for $(Y_s^\kappa)_{s\ge0}$ and $(Y_s^c)_{s\ge0}$ even positive, we have

$$\begin{aligned}&\mathbb{P}\left(Y_s^0=0\,\forall s\le t, N_s^\kappa=n_\kappa\,\forall\delta_1\le s\le t, N_s^c=0\,\forall s\le t-\delta_2, N_t^c=m_c\right)\\&=\mathbb{P}\left(Y_s^0=0\,\forall s\le t\right)\mathbb{P}\left(N_{\delta_1}^\kappa=n_\kappa\right)\mathbb{P}\left(N_s^\kappa=0\,\forall s\le t-\delta_1\right)\\&\qquad\cdot\mathbb{P}\left(N_s^c=0\,\forall s\le t-\delta_2\right)\mathbb{P}\left(N_{\delta_2}^c=m_c\right)>0.\end{aligned}$$

This finishes the proof of (c.2) and thus for the situation of (c). □

PROOF OF LEMMA 6.4. **Let us consider the case** $\sigma > 0$**.** For $\Pi_2 := \Pi$ and $\eta \in (0,1)$ the decomposition of (9) reads

$$X_t = -a_\eta t + \sigma B_t + P_t^\eta + M_t^\eta$$

with $a_\eta = a + \int_{(-1,1)\setminus(-\eta,\eta)} x\Pi(dx)$.

Now let $x \in (-\infty, K)$. Let $\varepsilon > 0$ be such that

$$\varepsilon < \min\{K - x, K\}.$$

Due to (10) we can choose $\eta > 0$ such that

$$\mathbb{P}\left(\sup_{s\le t} |M_s^\eta| \ge \frac{\varepsilon}{2}\right) < 1.$$

Further we have that $\mathbb{P}(P_s^\eta = 0\, \forall s \le t) > 0$. Let $f : [0,t] \to \mathbb{R}$ be defined by $f(s) := \frac{s}{t}x$. From the theory of Brownian motion we know that

$$\mathbb{P}\left(|\sigma B_s - a_\eta s - f(s)| < \frac{\varepsilon}{2}\, \forall s \in [0,t]\right) > 0,$$

for example see Theorem 38 in [25]. Note that

$$\begin{aligned}&\left\{|\sigma B_s - a_\eta s - f(s)| < \frac{\varepsilon}{2}\, \forall s \in [0,t]\right\}\\ &\subseteq \left\{|\sigma B_t - a_\eta t - x| < \frac{\varepsilon}{2}\right\} \cap \left\{\sup_{s\le t} |\sigma B_s - a_\eta s| < \frac{\varepsilon}{2} + \max\{0, x\}\right\}.\end{aligned}$$

This means

$$\begin{aligned}&\left\{|\sigma B_s - a_\eta s - f(s)| < \frac{\varepsilon}{2}\, \forall s \in [0,t]\right\} \cap \{P_s^\eta = 0\, \forall s \le t\} \cap \left\{\sup_{s\le t} |M_s^\eta| < \frac{\varepsilon}{2}\right\}\\ &\subseteq \{|X_t - x| < \varepsilon\} \cap \left\{\sup_{s\le t} X_s < K\right\}.\end{aligned}$$

This yields

$$\begin{aligned}0 &< \mathbb{P}\left(|\sigma B_s - a_\eta s - f(s)| < \frac{\varepsilon}{2}\, \forall s \le t\right) \mathbb{P}(P_s^\eta = 0\, \forall s \le t)\, \mathbb{P}\left(\sup_{s\le t} |M_s^\eta| < \frac{\varepsilon}{2}\right)\\ &\le \mathbb{P}\left(X_t \in (x - \varepsilon, x + \varepsilon), \sup_{s\le t} X_s < K\right) = \mathbb{P}(X_t \in (x - \varepsilon, x + \varepsilon), \tau_K > t).\end{aligned}$$

Thus we have $\operatorname{supp}(\mathbb{P}(X_t \in \cdot\,, \tau_K > t)) = (-\infty, K]$.

Now assume that $\sigma^2 = 0$. Define $\Pi_1(\mathrm{d}x) := x^2 \mathbb{1}_{(-1,1)}(x)\Pi(\mathrm{d}x)$ and $\Pi_2 := \Pi - \Pi_1$. Observe that it holds $\Pi_1(\mathbb{R}) < \infty$. For $\eta \in (0,1)$ the decomposition of (9) reads

$$X_t = Y_t - a_\eta t + P_t^\eta + M_t^\eta$$

with $a_1 := a + \int_{\mathbb{R}} x\Pi_1(\mathrm{d}x)$ and $a_\eta = a_1 + \int_{(-1,1)\setminus(-\eta,\eta)} x\Pi_2(dx)$.

In the following we distinguish the following three cases.

(I) $\Pi((-\infty,0)) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|)\Pi(\mathrm{d}x) = \infty$,
(II) $\Pi((0,\infty)) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|)\Pi(\mathrm{d}x) = \infty$,
(III) $0 \in \operatorname{supp}(\Pi)$ and $\Pi((-\infty,0)) > 0$ and $\Pi((0,\infty)) > 0$.

Note that these cases are exhausting for (i) and (ii) if $\sigma^2 = 0$.

Let $t > 0$ and $K > 0$. Let $x \in (-\infty, K)$ and $\varepsilon > 0$ such that

$$\varepsilon < \min\{K - x, K\}.$$

We first claim that for all three cases there exists $\eta > 0$ such that

$$\mathbb{P}\left(\sup_{s\le t} |M_s^\eta| \ge \frac{\varepsilon}{2}\right) < 1$$

and

$$\mathbb{P}\left(|Y_t - ta_\eta - x| < \frac{\varepsilon}{2}, \sup_{s\le t}(Y_s - sa_\eta) < K - \frac{\varepsilon}{2}\right) > 0.$$

Let us assume for the moment that the claim is true. We will finish the proof of the theorem from here and prove the claim further below. Recall that $X_t = Y_t - a_\eta t + P_t^\eta + M_t^\eta$, thus

$$\left\{|Y_t - ta_\eta - x| < \frac{\varepsilon}{2}, \sup_{s\le t}(Y_s - sa_\eta) < K - \frac{\varepsilon}{2}\right\} \cap \{P_s^\eta = 0\, \forall s \le t\} \cap \left\{\sup_{s\le t} |M_s^\eta| < \frac{\varepsilon}{2}\right\}$$

$$\subseteq \left\{|X_t - x| < \varepsilon, \sup_{s\le t} X_s < K\right\} = \{X_t \in (x - \varepsilon, x + \varepsilon), \tau_K > t\}.$$

Note that $\mathbb{P}\left(P_s^\eta = 0\, \forall s \le t\right) > 0$ and $\mathbb{P}\left(\sup_{s\le t} |M_s^\eta| < \frac{\varepsilon}{2}\right) > 0$ and by independence of the events we obtain

$$\mathbb{P}\left(X_t \in (x - \varepsilon, x + \varepsilon), \tau_K > t\right) > 0,$$

which implies the statement of the theorem, since $x \in (-\infty, K)$ was arbitrary.

Let us now prove the claim for every case separately.

Let us assume (I). This implies that for all $\eta > 0$ we have $\Pi((0, \eta)) = \infty$, hence

$$0 \in \operatorname{supp}(\Pi(\cdot \cap (0, \infty))).$$

Moreover, this implies that for $\eta \in (0, 1)$ we have $a_\eta = a_1 + \int_{[\eta,1)} x\Pi_2(dx)$ and

$$\int_{(0,1)} x\Pi_2(dx) = \infty,$$

which implies $\lim_{\eta\to 0} a_\eta = \infty$. With (10) in mind we can choose $\eta > 0$ such that

$$a_\eta t \ge \max\{-(x - \varepsilon/2), 0\} \quad \text{and} \quad \mathbb{P}\left(\sup_{s\le t} |M_s^\eta| \ge \frac{\varepsilon}{2}\right) < 1.$$

Observe that, since $0 \in \operatorname{supp}(\Pi_1(\cdot \cap (0, \infty)))$ and $a_\eta \ge 0$, the condition (a) of Lemma 6.5 is fulfilled for the process $\tilde{X}_t = Y_t - a_\eta t$, thus we have that

$$\operatorname{supp}\left(\mathbb{P}\left(Y_t - a_\eta t \in \cdot\, , \sup_{s\le t}(Y_s - a_\eta s) < K - \frac{\varepsilon}{2}\right)\right) = \left(-a_\eta t, K - \frac{\varepsilon}{2}\right],$$

which implies the assertion of the claim. This finishes the proof for the case (I).

Let us assume (II). Then for all $\eta > 0$ we have $\Pi((-\eta, 0)) = \infty$, hence

$$0 \in \operatorname{supp}(\Pi(\cdot \cap (-\infty, 0))).$$

Moreover, this implies that for $\eta \in (0, 1)$ we have $a_\eta = a_1 + \int_{(1,-\eta]} x\Pi_2(dx)$ and

$$\int_{(-1,0)} x\Pi_2(dx) = -\infty,$$

which implies $\lim_{\eta\to 0} a_\eta = -\infty$. With (10) in mind we can choose $\eta > 0$ such that

$$-a_\eta t \geq K - \frac{\varepsilon}{2} \quad \text{and} \quad \mathbb{P}\left(\sup_{s\leq t} |M_s^\eta| \geq \frac{\varepsilon}{2}\right) < 1.$$

Observe that now condition (b) of Lemma 6.5 is fulfilled for the process $\tilde{X}_t = Y_t - a_\eta t$, thus we have that

$$\text{supp}\left(\mathbb{P}\left(Y_t - a_\eta t \in \cdot\,, \sup_{s\leq t}(Y_s - a_\eta s) < K - \frac{\varepsilon}{2}\right)\right) = \left(-\infty, K - \frac{\varepsilon}{2}\right],$$

which implies the assertion of the claim. This finishes the proof for the case (II).

Let us assume (III). Due to (10) we can choose $\eta > 0$ such that

$$\mathbb{P}\left(\sup_{s\leq t} |M_s^\eta| \geq \frac{\varepsilon}{2}\right) < 1.$$

The process $\tilde{X}_t = Y_t - a_\eta t$ inherits the properties of (ii) and fulfills the conditions of (c) of Lemma 6.5. Thus we have that

$$\text{supp}\left(\mathbb{P}\left(Y_t - a_\eta t \in \cdot\,, \sup_{s\leq t}(Y_s - a_\eta s) < K - \frac{\varepsilon}{2}\right)\right) = \left(-\infty, K - \frac{\varepsilon}{2}\right].$$

This implies the assertion of the claim. This finishes the proof for the case (III). □

PROOF OF PROPOSITION 6.2. **Regarding (III)** note that by translation of the starting point Lemma 6.4 yields that for any $t > 0$ and for any $K > x > 0$ we have that

$$\text{supp}(\mathbb{P}_x(X_t \in \cdot, \tau_K > t)) = (-\infty, K].$$

Thus we obtain by Lemma 6.3 that for any boundary function $b : [0,\infty] \to [-\infty,\infty]$ and $t > 0$ with $\mathbb{P}(\tau_b > t) > 0$ it holds

$$\overline{\text{supp}}(\mathbb{P}(X_t \in \cdot\,|\tau_b > t)) = (-\infty, b(t)].$$

This finishes the proof for (3).

Let us prove (I) and (II): For $-1 < \kappa_1 < \kappa_2 < 1$ define $\Pi_1(\mathrm{d}x) := |x|\mathbb{1}_{(\kappa_1,\kappa_2)}(x)\Pi(\mathrm{d}x)$ and $\Pi_2 := \Pi - \Pi_1$. For $\eta \in (0,1)$ the decomposition of (9) reads

$$X_t = Y_t + P_t^\eta + S_t^\eta,$$

where $S_t^\eta := M_t^\eta - a_\eta t$ and since there is no drift, i.e. $a_\eta = \int_{(-\eta,\eta)} x\Pi_2(\mathrm{d}x)$,

$$-\log\left(\mathbb{E}\left[e^{i\theta S_1^\eta}\right]\right) = \int_{(-\eta,\eta)} (1 - e^{i\theta x})\Pi_2(\mathrm{d}x).$$

Let us denote

$$\tau_b^\eta := \inf\{t > 0 : S_t^\eta \geq b(t)\}.$$

Since there is no drift, in both of the cases (1) or (2) the process $(|S_t^\eta|)_{t\geq 0}$ is a subordinator with $\mathbb{E}[S_t^\eta] = t\int_{(-\eta,\eta)} |x|\Pi_2(dx)$. This means that by Markov's inequality for every $C > 0$ we can choose $\eta > 0$ such that

$$\mathbb{P}\left(\sup_{s\leq t} |S_s^\eta| \geq C\right) = \mathbb{P}(|S_t^\eta| \geq C) \leq \frac{t}{C}\int_{(-\eta,\eta)} |x|\Pi_2(dx) < 1. \tag{11}$$

For $\kappa_1 < \kappa_2$ we write

$$Y_t = \int_0^t (Y_s - Y_{s-})\,\mathrm{d}N_s^\kappa \quad \text{with} \quad N_t^\kappa := \sum_{s\le t} \mathbb{1}_{\mathbb{R}\setminus\{0\}}(Y_s - Y_{s-}),$$

and N_t^κ is a Poisson process with rate $\Pi_1((\kappa_1,\kappa_2)) \ge 0$.

Now we treat (1) and (2) separately.

Assume the conditions of (I). Assume that $\mathbb{P}(\tau_b > t) > 0$ for $t > 0$. We have $\mathrm{supp}(\Pi) \subseteq [0,\infty)$. Thus we have $X_r \ge N_r^\eta$ almost surely. This implies

$$0 < \mathbb{P}(\tau_b > t) \le \mathbb{P}\left(\tau_b^\eta > t\right).$$

Let $\delta \in (0,t)$ and define

$$K_\delta := \inf_{s\in[t-\delta,t]} b(s) > 0.$$

Let $x \in (0, K_\delta)$ and $0 < \varepsilon < \min(x, K_\delta - x)$. By (11) choose $\eta \in (0,1)$ such that

$$\mathbb{P}\left(\sup_{s\le t} |S_s^\eta| \ge \frac{2\varepsilon}{3}\right) < \mathbb{P}(\tau_b > t).$$

Since $\tau_b^\eta \ge \tau_b$ almost surely, we have that

$$\begin{aligned}\mathbb{P}\left(\sup_{s\le t}|S_s^\eta| < \frac{2\varepsilon}{3}, \tau_b^\eta > t\right) &\ge \mathbb{P}\left(\sup_{s\le t}|S_s^\eta| < \frac{2\varepsilon}{3}, \tau_b > t\right)\\ &\ge \mathbb{P}(\tau_b > t) - \mathbb{P}\left(\sup_{s\le t}|S_s^\eta| \ge \frac{2\varepsilon}{3}\right) > 0.\end{aligned}$$

Further, since $0 \in \mathrm{supp}(\Pi) \subseteq [0,\infty)$ we have that there is $\kappa \in \mathrm{supp}(\Pi) \cap (0, \varepsilon/3)$. Since $\kappa < \varepsilon/3$ there is $n_\kappa \in \mathbb{N}$ and $0 < \kappa_1 < \kappa < \kappa_2 < \varepsilon/3$ such that

$$(\kappa_1,\kappa_2)\cdot n_\kappa \subseteq \left(x - \frac{\varepsilon}{3}, x + \frac{\varepsilon}{3}\right).$$

Now observe, since $(Y_s)_{s\ge 0}$ only has jumps of size contained in (κ_1,κ_2), that

$$\begin{aligned}&\left\{\sup_{s\le t}|S_s^\eta| < \frac{2\varepsilon}{3}, \tau_b^\eta > t\right\} \cap \{P_s^\eta = 0\,\forall s \le t\} \cap \{N_s^\kappa = 0\,\forall s \le t-\delta\} \cap \{N_t^\kappa = n_\kappa\}\\ &\subseteq \left\{\sup_{s\le t}|S_s^\eta| < \frac{2\varepsilon}{3}\right\} \cap \{\tau_b > t-\delta\} \cap \{P_s^\eta = 0\,\forall s\le t\}\\ &\qquad \cap \left\{Y_t \in \left(x - \frac{\varepsilon}{3}, x + \frac{\varepsilon}{3}\right)\right\}\\ &\subseteq \{\tau_b > t-\delta\} \cap \{\tau_{K_\delta} \notin [t-\delta, t]\} \cap \{X_t \in (x-\varepsilon, x+\varepsilon)\}\\ &\subseteq \{\tau_b > t\} \cap \{X_t \in (x-\varepsilon, x+\varepsilon)\}.\end{aligned}$$

By independence and the Markov property the event on the left-hand-side has positive probability and thus we obtain

$$(0, K_\delta) \subseteq \mathrm{supp}(\mathbb{P}(X_t \in \cdot\,, \tau_b > t)).$$

Since $K_\delta \to b(t)$ for $\delta \to 0$, we obtain that

$$\overline{\mathrm{supp}}(\mathbb{P}(X_t \in \cdot\,, \tau_b > t)) = [0, b(t)].$$

Assume the conditions of (II). Let $x \in (-\infty, b(t))$ and $\varepsilon \in (0, b(t) - x)$. By (11) choose $\eta \in (0,1)$ such that

$$\mathbb{P}\left(\sup_{s\le t} |S_s^\eta| < \frac{2\varepsilon}{3}\right) > 0.$$

Assume $\mathbb{P}(\tau_b > 0) > 0$. By Blumenthal's law we have that $\tau_b > 0$ almost surely. Now, since $\Pi_1(\mathbb{R}) < \infty$ and $\Pi_2(\mathbb{R} \setminus (-\eta, \eta)) < \infty$, we have

$$\begin{aligned}
0 &< \mathbb{P}(Y_s = 0\, \forall s \le t, P_s^\eta = 0\, \forall s \le t) = \mathbb{P}(Y_s = 0\, \forall s \le t, P_s^\eta = 0\, \forall s \le t, \tau_b > 0) \\
&= \mathbb{P}\left(Y_s = 0\, \forall s \le t, P_s^\eta = 0\, \forall s \le t, \tau_b^\eta > 0\right) \\
&= \mathbb{P}(Y_s = 0\, \forall s \le t, P_s^\eta = 0\, \forall s \le t)\, \mathbb{P}\left(\tau_b^\eta > 0\right).
\end{aligned}$$

Consequently we have $\tau_b^\eta > 0$ almost surely. Hence there is $\delta \in (0, t)$ such that

$$\mathbb{P}\left(\sup_{s\le t} |S_s^\eta| < \frac{2\varepsilon}{3}, \tau_b^\eta > \delta\right) > 0.$$

Now since $0 \in \operatorname{supp}(\Pi) \cap (-\infty, 0)$ there is $\kappa \in \operatorname{supp}(\Pi) \cap (-\varepsilon, 0)$. Since $|\kappa| < \varepsilon$ there is $n_\kappa \in \mathbb{N}$ and $-\varepsilon < \kappa_1 < \kappa < \kappa_2 < 0$ such that

$$(\kappa_1, \kappa_2) \cdot n_\kappa \subseteq \left(x - \frac{\varepsilon}{3}, x + \frac{\varepsilon}{3}\right).$$

Since in the situation of (II) we have $\operatorname{supp}(\Pi) \subseteq (-\infty, 0]$, it follows that $X_r \le S_r^\eta$. Hence also $\tau_b \ge \tau_b^\eta$. Therefore, with using that $b(t) \le b(u)$ for all $u \in [0, t]$, we have

$$\begin{aligned}
&\left\{\sup_{s\le t} |S_s^\eta| < \frac{2\varepsilon}{3}, \tau_b^\eta > \delta\right\} \cap \{P_s^\eta = 0\, \forall s \le t\} \cap \{N_\delta^\kappa = n_\kappa\} \cap \{N_\delta^\kappa = n_\kappa\, \forall s \in [\delta, t]\} \\
&\subseteq \left\{\sup_{s\le t} |S_s^\eta| < \frac{2\varepsilon}{3}\right\} \cap \{\tau_b > \delta\} \cap \{P_s^\eta = 0\, \forall s \le t\} \\
&\qquad \cap \left\{Y_s^\kappa < b(t) - \frac{2\varepsilon}{3}\, \forall s \in [\delta, t]\right\} \cap \left\{Y_t \in \left(x - \frac{\varepsilon}{3}, x + \frac{\varepsilon}{3}\right)\right\} \\
&\subseteq \{\tau_b > \delta\} \cap \{\tau_b \notin [\delta, t]\} \cap \{X_t \in (x - \varepsilon, x + \varepsilon)\} \\
&= \{\tau_b > t\} \cap \{X_t \in (x - \varepsilon, x + \varepsilon)\}.
\end{aligned}$$

By independence and the Markov property the event on the left-hand-side has positive probability and thus we obtain

$$(-\infty, b(t)) \subseteq \operatorname{supp}(\mathbb{P}(X_t \in \cdot\,, \tau_b > t)).$$

This finishes the proof. □

PROOF OF THEOREM 2.13. A Lévy process has right-continuous paths by definition. Furthermore, as a càdlàg Feller-process a Lévy process is quasi-left-continuous and a Markov process, see Proposition 7 and Proposition 6 of [10]. This gives (E2) and (U1). For the order-preservation, note that, by Theorem 1.A.1 of [44], we have $\mu_1 \preceq_{\mathrm{st}} \mu_2$ if and only if there exist random variables $Z_i \sim \mu_i$ such that $Z_1 \le Z_2$. We can choose them independently from $(X_t)_{t\ge0}$, hence we have that $Z_1 + X_t \le Z_2 + X_t$ and $\mathbb{P}_0(Z_i + X_t \in \cdot\,) = \mathbb{P}_\mu(X_t \in \cdot\,)$. By Theorem 1.A.1 of [44] it follows that $\mathbb{P}_{\mu_1}(X_t \in \cdot) \preceq_{\mathrm{st}} \mathbb{P}_{\mu_2}(X_t \in \cdot)$. This gives (U2).
Existence: By Proposition 6.1 we obtain that ((E1) ⇒ (E3)). Assuming (E1) therefore implies that the conditions of Theorem 2.3 are fulfilled, and thus a solution for the inverse

first-passage time problem exists if (E1) holds.
Uniqueness: It is left to show that in the situation of (a) or (b) we have (U3) for the corresponding $I^\xi \subset (0,t^\xi)$. Let b be a boundary function with $\tau_b \overset{\mathrm{d}}{=} \xi$. We want to apply Proposition 6.2.
Assume (a): Let $t \in I^\xi = (0,t^\xi)$. We can exhaust (a) by the case distinction

(a.i) $(X_t)_{t\geq 0}$ has unbounded variation,
(a.ii.1) $0 \in \operatorname{supp}(\Pi)$ and $\Pi((-\infty,0)) > 0$ and $\Pi((0,\infty)) > 0$,
(a.ii.2') $0 \in \operatorname{supp}(\Pi)$ and $X_t = Y_t + \gamma t$, where $(Y_t)_{t\geq 0}$ is a subordinator without drift and $\gamma \in \mathbb{R}$.

Note that the case (a.ii.2') can be rephrased as the case that $0 \in \operatorname{supp}(\Pi)$ and $\Pi((-\infty,0)) = 0$ and $(X_t)_{t\geq 0}$ has bounded variation. Observe, if $x \in \mathbb{R}$, then by a translation according to x and Proposition 6.2 we have for $t > 0$ with $\mathbb{P}_x(\tau_b > t) > 0$ (for (a.ii.2') this implies $x + \gamma t < b(t)$) that

$$\overline{\operatorname{supp}}(\mathbb{P}_x(X_t \in \cdot\ |\tau_b > t)) = \begin{cases} (-\infty, b(t)] & : \text{(a.i) or (a.ii.1)} \\ [x+\gamma t, b(t)] & : \text{(a.ii.2').} \end{cases}$$

Due to $t \in (0,t^\xi)$ we have $\mathbb{P}(\tau_b > t) > 0$. Since $\mathbb{P} = \int_{\mathbb{R}} \mathbb{P}_x \mu(\mathrm{d}x)$, we obtain

$$\sup \operatorname{supp}(\mathbb{P}_\mu(X_t \in \cdot\ |\tau_b > t)) = b(t).$$

Assume (b): Let $t \in I^\xi = \operatorname{supp}(\mathbb{P}(\xi \in \cdot)) \cap (0,t^\xi)$. The case that $(X_t)_{t\geq 0}$ has unbounded variation is already covered by (a). Let us therefore assume that $(X_t)_{t\geq 0}$ has bounded variation. This implies that $X_t = \tilde{X}_t + \gamma t$, where $(-\tilde{X}_t)_{t\geq 0}$ is a subordinator without drift. Without loss of generality we can assume that $\gamma = 0$ by considering the process $(\tilde{X}_t)_{t\geq 0}$ and the boundary $\tilde{b}(t) = b(t) - \gamma t$ instead. Hence from now on we assume that $(-X_t)_{t\geq 0}$ is a subordinator without drift and $0 \in \operatorname{supp}(\Pi)$. Suppose there is $0 \leq u < t$ such that $b(t) > b(u)$. For $\varepsilon \in (0, b(t) - b(u))$ there exists $\delta \in (0, t-u)$ such that

$$\inf_{s \in [t-\delta, t+\delta]} b(s) \geq b(t) - \varepsilon > b(u).$$

But this implies, since $(X_t)_{t\geq 0}$ has non-increasing paths, that

$$\begin{aligned} 0 < \mathbb{P}(\xi \in (t-\delta,t+\delta)) &= \mathbb{P}(\tau_b \in (t-\delta,t+\delta)) = \mathbb{P}(\tau_b \in (t-\delta,t+\delta), \tau_b > u) \\ &\leq \mathbb{P}(\exists s \in (t-\delta,t+\delta) : X_s \geq b(u), \forall s \geq u : X_s < b(u)) = 0. \end{aligned}$$

This contradiction shows that $b(t) \leq b(u)$ for all $u \leq t$. For $x \in \mathbb{R}$ with $\mathbb{P}_x(\tau_b > 0) > 0$, by Proposition 6.2, we get that

$$\operatorname{supp}(\mathbb{P}_x(X_t \in \cdot\ |\tau_b > t)) = (-\infty, \min(x, b(t))]$$

Since $(X_t)_{t\geq 0}$ has non-increasing paths and $t \in \operatorname{supp}(\mathbb{P}(\tau_b \in \cdot))$ we have

$$\mu(\{x \geq b(t) : \mathbb{P}_x(\tau_b > 0) > 0\}) > 0.$$

This implies that

$$\sup \operatorname{supp}(\mathbb{P}_\mu(X_t \in \cdot\ |\tau_b > t)) = b(t).$$

Therefore, under the assumptions that (E1) and ((a) or (b)) are fulfilled, and hence, by Theorem 2.8, the boundary function b with $\tau_b \overset{\mathrm{d}}{=} \xi$ is unique on I^ξ. □

7. Conditions for diffusions on an interval. In this section we establish conditions under which a diffusion process in an interval, which satisfies a stochastic differential equation up to an explosion time, fulfills the assumptions required for existence and uniqueness of solutions in the inverse first-passage time problem. The proof of Theorem 2.16 is to be found at the end of the section. At first, we will collect the essential steps in preliminary statements. Let $(X_t)_{t\geq 0}$ be a diffusion on an interval E according to Definition 2.15.

PROPOSITION 7.1. *Assume that $R \notin E$. Further, assume that $\sigma \in C^1((L,R))$, $\sigma > 0$ and that β is locally bounded on (L,R). Let $x \in E$. Let $b : [0,\infty) \to [-\infty,\infty]$ be a boundary function. It holds that*

$$\mathbb{P}_x\left(\tau_b = \tau_b'\right) = 1.$$

In the case of Brownian motion the following statement was proved in Proposition 6.1 in [22].

PROPOSITION 7.2. *Let $b : [0,\infty) \to \overline{E}$ be a boundary function. Assume that $\sigma \in C^1((L,R))$, $\sigma > 0$ and that β is locally bounded on (L,R). Let μ be a probability measure on E. Assume that $\mathbb{P}_\mu(\tau_b > 0) > 0$ and that $\mathbb{P}_\mu(X_t \in \cdot)$ is diffuse for every $t > 0$. Then*

$$\operatorname{supp}(\mathbb{P}_x(X_t \in \cdot \;|\tau_b > t)) = [L, b(t)]$$

for every $t < \inf\{s > 0 : b(s) = L\}$.

The first step towards Proposition 7.1 will be the following.

LEMMA 7.3. *Let $b : [0,\infty) \to \overline{E}$ be a boundary function. Assume that $\sigma \in C^1((L,R))$, $\sigma > 0$ and that β is locally bounded. Then for $x \in (L,R)$ we have*

$$\mathbb{P}_x\left(\tau_b < \tau_b' \wedge S\right) = 0,$$

where $S := \lim_{n\to\infty} S_n$, where S_n are defined in Definition 2.15 (iii).

The idea is to reduce the situation to Brownian motion and use the fact that for Brownian motion the desired statements are already known. For example, Proposition 2 in [13] and Lemma 6.2 in [22] prove that for Brownian motion it holds $\tau_b = \tau_b'$ almost surely. For this we follow the idea of [13] from Proposition 2 therein. We first scale the process in the spatial coordinate as in (4.2) in [13] and then change the measure by using the Girsanov theorem.

For simplicity we assume that $\sigma \in C^1((L,R))$, $\sigma > 0$ and that β is locally bounded. Let $c \in (L,R)$ be fixed and for $x \in (L,R)$ define

$$f(x) := \int_c^x \frac{1}{\sigma(z)}\,\mathrm{d}z. \tag{12}$$

We have $f \in C^2(L,R)$ and that f is strictly increasing and invertible. Let $n \in \mathbb{N}$. Under $\mathbb{P}_x$ the process $(X_{t\wedge S_n})_{t\geq 0}$ is a semimartingale and due to the Itô formula it follows that

$$f(X_{t\wedge S_n}) = f(X_0) + \int_0^{t\wedge S_n}\left(\frac{\beta(X_s)}{\sigma(X_s)} - \frac{1}{2}\sigma'(X_s)\right)\mathrm{d}s + B_{t\wedge S_n}.$$

This means that the process given by $Y_t := f(X_t)$ fulfills

$$Y_{t\wedge S_n} = Y_0 + \int_0^{t\wedge S_n} \tilde{\beta}(Y_s)\,\mathrm{d}s + B_{t\wedge S_n},$$

where

$$\tilde{\beta}(y) := \frac{\beta(f^{-1}(y))}{\sigma(f^{-1}(y))} - \frac{1}{2}\sigma'(f^{-1}(y)).$$

Note that $\tilde{\beta}(Y_t)$ is uniformly bounded in $t \le S_n$. Let $T > 0$ be fixed. Then

$$U_t := \exp\left(-\int_0^{t\wedge T\wedge S_n} \tilde{\beta}(Y_s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^{t\wedge T\wedge S_n} (\tilde{\beta}(Y_s))^2\,\mathrm{d}s\right)$$

defines a uniformly integrable positive martingale. By Girsanov the measure

$$\mathrm{d}\tilde{\mathbb{P}}_x^{n,T} := U_T\,\mathrm{d}\mathbb{P}_x \tag{13}$$

defined on $\mathcal{F}_{T\wedge S_n}$ is equivalent to $\mathbb{P}_x$ on $\mathcal{F}_{T\wedge S_n}$ and $(Y_{t\wedge T\wedge S_n})_{t\ge 0}$ is a local martingale. Since its quadratic variation is $(t\wedge T\wedge S_n)_{t\ge 0}$, Lévy's characterization of Brownian motion shows that $(Y_{t\wedge T\wedge S_n})_{t\ge 0}$ is a Brownian motion stopped at $T\wedge S_n$.

PROOF OF LEMMA 7.3. Recall f from (12) and consider $Y_t = f(X_t)$. Define $\tilde{b} := f(b)$, where we allow $f(R) \in (-\infty, \infty]$ and $f(L) \in [-\infty, \infty)$. Note that

$$\tau_b = \inf\{t > 0 : Y_t \ge \tilde{b}(t)\}, \quad \tau_b' = \inf\{t > 0 : Y_t > \tilde{b}(t)\}.$$

Since under $\tilde{\mathbb{P}}_x^{n,T}$ from (13) the stopped process $(Y_{t\wedge T\wedge S_n})_{t\ge 0}$ is a stopped Brownian motion we have that

$$\tilde{\mathbb{P}}_x^{n,T}\left(\tau_b < \tau_b' \wedge T \wedge S_n\right) = 0.$$

Due to the equivalence of the measures $\tilde{\mathbb{P}}_x^{n,t}$ and $\mathbb{P}_x$ it follows that

$$\mathbb{P}_x\left(\tau_b < \tau_b' \wedge T \wedge S_n\right) = 0.$$

By first letting $T \to \infty$ and then $n \to \infty$ it follows that

$$\mathbb{P}_x\left(\tau_b < \tau_b' \wedge S\right) = 0.$$

This finishes the proof. □

PROOF OF PROPOSITION 7.1. The idea of this proof is to split the path into suitable excursions away from the lower boundary and to apply Lemma 7.3 for every excursion. We begin with assumptions by which we do not lose generality in order to reduce the complexity of the boundary involved.

Since $R \notin E$ we can assume that b takes values in $\overline{E}$. Due to the a.s. convergence of $\lim_{s\searrow 0} \tau_{b|_s} = \tau_b$ and $\lim_{s\searrow 0} \tau'_{b|_s} = \tau_b'$ we can assume that there is $s > 0$ such that $b(t) = R$ for $t < s$. Furthermore, we have $\tau_b \le \tau_b' \le \inf\{t > 0 : b(t) = L\}$. If $\tau_b < \tau_b'$ we have consequently $\tau_b < \inf\{t > 0 : b(t) = L\}$. We will therefore assume that there is $u < \inf\{t > 0 : b(t) = L\}$ such that $b(t) = R$ for all $t > u$ and $b(t) > L$ for all $t \in [s, u]$. Thus, using the lower semicontinuity, we now treat the case that $b(t) = R$ for $t \notin [s, u]$ and

$$L < \inf_{t\in[s,u]} b(t) \le \sup_{t\in[s,u]} b(t) \le R.$$

For $x \in (L, R)$ let us define

$$T_x := \inf\{t \ge 0 : X_t \le x\}.$$

Let $x_\ell, x_r \in \{\ell_n : n \in \mathbb{N}\}$ such that $L < x_\ell < x_r < \inf_{t\in[s,u]} b(t)$. For $k \in \mathbb{N}$ let us inductively define $\rho_0 := 0$,

$$\lambda_k := \inf\{t \ge \rho_{k-1} : X_t \le x_\ell\}$$

and

$$\rho_k := \inf\{t \geq \sigma_k : X_t \geq x_r\}.$$

Since $X_t \leq x_r < \inf_{t\in[s,u]} b(t)$ for all $t \in [\lambda_k, \rho_k]$ and all $k \in \mathbb{N}$ we have that, if $\tau_b \neq \tau_b'$, then there is $k \in \mathbb{N}_0$ such that

$$\rho_k < \infty \quad \text{and} \quad \tau_b \in (\rho_k, \lambda_k) \quad \text{and} \quad \tau_b < \tau_b'.$$

For $x \in E$ we have that

$$\begin{aligned}&\mathbb{P}_x\left(\tau_b \in (\rho_k, \lambda_k), \tau_b < \tau_b', \rho_k < \infty\right)\\&= \mathbb{E}_x\left[\mathbb{P}_{X_{\rho_k}}\left(\tau_{b^\theta} \in (0, T_{x_\ell}), \tau_{b^\theta} < \tau_{b^\theta}'\right)_{\theta=\rho_k} \mathbb{1}_{\{\rho_k<\infty\}}\mathbb{1}_{\{\rho_k<\tau_b\}}\right],\end{aligned}$$

where $b^\theta(t) = b(\theta + t)$. For the moment fix $\theta \geq 0$ and note that b^θ is a boundary function taking values in $\overline{E}$. Since $(X_t)_{t\geq 0}$ has continuous paths and $R \notin E$ we have $\lim_{n\to\infty} \tau_{r_n} = \infty$, where $(r_n)_{n\in\mathbb{N}}$ is the sequence from Definition 2.15 (iii). Therefore we have that $T_{x_\ell} \leq S = \lim_{n\to\infty} S_n$ and therefore for $z \in (L, R)$, by using Lemma 7.3, it holds

$$\mathbb{P}_z\left(\tau_{b^\theta} \in (0, T_{x_\ell}), \tau_{b^\theta} < \tau_{b^\theta}'\right) \leq \mathbb{P}_z\left(\tau_{b^\theta} < \tau_{b^\theta}' \wedge S\right) = 0.$$

Using this for $z = X_{\rho_k} \in \{X_0, x_r\} \subset (L, R)$ and plugging it back into the expectation above we have therefore

$$\mathbb{P}_x\left(\tau_b \in (\rho_k, \lambda_k), \tau_b < \tau_b', \rho_k < \infty\right) = 0.$$

This implies

$$\mathbb{P}_x\left(\tau_b \neq \tau_b'\right) \leq \sum_{k\in\mathbb{N}_0} \mathbb{P}_x\left(\tau_b \in (\rho_k, \lambda_k), \tau_b < \tau_b', \rho_k < \infty\right) = 0.$$

This finishes the proof of the first part. □

PROOF OF PROPOSITION 7.2. Let $t < \inf\{s > 0 : b(s) = L\}$ and $z \in (L, b(t))$ and $\varepsilon > 0$ such that $[z-\varepsilon, z+\varepsilon] \subseteq (L, b(t))$. Since $\mathbb{P}_\mu(\tau_b > 0) > 0$ there is $s \in (0,t)$ with $\mathbb{P}_\mu(\tau_b > s) > 0$. And since $\mathbb{P}_\mu(X_s \in \cdot)$ is diffuse there is $y \in (L, b(s))$ such that $y \in \operatorname{supp}(\mathbb{P}_\mu(X_s \in \cdot\,, \tau_b > s))$. Set $b^s(t) = b(s+t)$. Due to

$$\begin{aligned}&\mathbb{P}_\mu\left(X_t \in (z-\varepsilon, z+\varepsilon), \tau_b > t\right)\\&\geq \int_{\mathbb{R}} \mathbb{P}_u\left(X_{t-s} \in (z-\varepsilon, z+\varepsilon), \tau_{b^s} > t-s\right)\mathbb{P}_\mu\left(X_s \in \mathrm{d}u, \tau_b > s\right)\end{aligned}$$

it suffices to show that we have

$$\mathbb{P}_u\left(X_{t-s} \in (z-\varepsilon, z+\varepsilon), \tau_{b^s} > t-s\right) > 0 \tag{14}$$

for $u \in U$, where $U \subseteq (L, b(s))$ is a neighborhood of y. For this, let $n \in \mathbb{N}$ be large enough such that

$$y \in (\ell_n, r_n), \quad (z-\varepsilon, z+\varepsilon) \subseteq (\ell_n, r_n).$$

Further, choose $T > t - s$. Recall f from (12). Let $a : [0, t-s] \to \mathbb{R}$ be a continuous function and $\delta > 0$ such that

- $a(0) = f(y)$ and $(a(r) - \delta, a(r) + \delta) \subseteq (f(\ell_n), f(r_n))$ for all $r \in [0, t-s]$,
- $(a(t-s) - \delta, a(t-s) + \delta) \subseteq (f(z-\varepsilon), f(z+\varepsilon))$,
- $f(a(r) + \delta) < f(b(s+r))$ for all $r \in [0, t-s]$.

An explicit construction of the function a can be made as in Lemma 2.3.6 of [33]. The last point is possible since $b(sr) > L$ for all $r \in [0, t-s]$ and b is lower semicontinuous. Recall $Y_r = f(X_r)$ and $\tilde{\mathbb{P}}_u^{n,T}$ from (13) and that under $\tilde{\mathbb{P}}_u^{n,T}$ the stopped process $(Y_{r\wedge T\wedge S_n})_{r\geq 0}$ is a stopped Brownian motion. Note that it happens with positive probability that a Brownian motion started at $f(u) \in (f(y)-\delta, f(y)+\delta)$ stays up to time $t-s$ in a tube which follows a continuous function, for instance see Theorem 38 of [25]. This argument was already used in Proposition 3.1 of [22] in case of Brownian motion. This leads to

$$\begin{aligned}
&\tilde{\mathbb{P}}_u^{n,T}\left(Y_{t-s} \in (f(z-\varepsilon), f(z+\varepsilon)), \tau_{b^s} > t-s, S_n > t-s\right) \\
&\geq \tilde{\mathbb{P}}_u^{n,T}\left(|Y_r - a(r)| < \delta \,\forall r \in [0, t-s]\right) > 0.
\end{aligned}$$

Due to the equivalence of the measures $\tilde{\mathbb{P}}_x^{n,T}$ and $\mathbb{P}_x$ we obtain that (14) is true. All in all, since z was arbitrary, this means that

$$\operatorname{supp}(\mathbb{P}_x\left(X_t \in \cdot \;|\tau_b > t\right)) = [L, b(t)].$$

This finishes the proof of the statement. □

PROOF OF THEOREM 2.16. By our definition of a diffusion on an interval we already assumed that the process has continuous paths and is a strong Markov process. This gives (E2) and (U1). Since $(X_t)_{t\geq 0}$ is a strong Markov process and has continuous paths, the transition probabilities preserve the usual stochastic order, since paths that started from different positions can be let run together after they have met, for details see Lemma A.6.1 in [33]. Hence we have (U2). Moreover, due to the assumptions on the coefficients and that $R \notin E$ we can apply Proposition 7.1 and obtain (E3).

Now assume that (E1) holds. Let $t \in (0, t^\xi)$ and assume that $\tau_b \overset{\mathrm{d}}{=} \xi$. Since $\mathbb{P}_\mu\left(\tau_b > t\right) > 0$ we have that $t < \inf\{s > 0 : b(s) = L\}$. Therefore, Proposotion 7.2 implies that

$$\sup\operatorname{supp}(\mathbb{P}_\mu\left(X_t \in \cdot\,, \tau_b > t\right)) = b(t).$$

This gives (U3), and therefore there exists a boundary function b with $\tau_b \overset{\mathrm{d}}{=} \xi$ is unique on $(0, t^\xi)$. □

APPENDIX

The following statement follows from Theorem 6 in [46], but for completeness we give an own proof, which makes use of probabilistic arguments.

LEMMA A.4. *Let $b : [0,\infty] \to [-\infty, \infty]$ be an arbitrary function. Then the set*

$$\left\{t \in (0,\infty) : \max\left(\liminf_{s\nearrow t} b(s), \liminf_{s\searrow t} b(s)\right) > b(t)\right\}$$

is countable.

PROOF. We only consider the set

$$S_b := \left\{t \in (0,\infty) : \liminf_{s\nearrow t} b(s) > b(t)\right\}$$

since then the statement follows for the remaining points by consideration of the map $(0,\infty) \ni t \mapsto b(1/t)$.

Further, let

$$\varphi : [-\infty,\infty] \to [-1,1],\; \varphi(x) := \frac{x}{1+|x|}\mathbb{1}_{\mathbb{R}}(x) + \operatorname{sgn}(x)\mathbb{1}_{\{-\infty,\infty\}}(x)$$

and set $\tilde{b}(t) := \varphi(b(t))$. Since $S_b \subseteq S_{\tilde{b}}$ we can assume that b takes values in $[-1,1]$.

The function defined by

$$b^*(t) := \min\left(\liminf_{s\to t} b(s), b(t)\right)$$

is lower semicontinuous and it holds $S_b \subseteq S_{b^*}$. Thus without loss of generality we can assume that b is a lower semicontinuous function.

Let $t \in S_b$. Then there is $\varepsilon > 0$ such that there exists $\delta > 0$ with

$$b(s) \geq b(t) + \varepsilon \quad \forall s \in (t-\delta, t).$$

Let $(B_t)_{t\geq 0}$ be a Brownian motion starting from a deterministic point $B_0 = x < -1$. For a function $f : [0,\infty] \to \mathbb{R}$ define

$$\tau_f := \inf\{s > 0 : B_s \geq f(s)\}.$$

Let $K := b(t) + \varepsilon$. Then we have

$$\begin{aligned}\mathbb{P}(\tau_b = t) &\geq \mathbb{P}(\tau_{-1} > t-\delta, \tau_K > t, B_t \in (b(t), b(t)+\varepsilon)) \\ &= \mathbb{P}\left(\sup_{s\in[0,t-\delta]} B_s < -1, \sup_{s\in[t-\delta,t]} B_s < K, B_t \in (b(t), b(t)+\varepsilon)\right) > 0.\end{aligned}$$

Therefore, we have

$$S_b \subseteq \{t \in (0,\infty) : \mathbb{P}(\tau_b = t) > 0\},$$

where the right-hand side is a countable set. This finishes the proof. □

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